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Research article

On multivalued maps for φ-contractions involving orbits with application

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Abstract: In [14], Proinov established the existence of fixed point theorems regarding as a generalization of the Banach contraction principle (BCP) of self mapping under an influence of gauge function (GF). In this paper, we develop some existence results on φ-contraction for multivalued maps via b-Bianchini-Grandolfi gauge function (B-GGF) in class of b-metric spaces and consequently assure the existence results in the module of simulation function as well α-admissible mapping. An extensive set of nontrivial example is given to justify our claim. At the end, we give an application to prove the existence behavior for the system of integral inclusion.

Keywords: fixed point; b-Bianchini-Grandolfi gauge function; simulation function; b-metric space
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1. Introduction and preliminaries

Let $(\hat{U}, d)$ be a metric space. For $\widehat{\ell} \in \hat{U}$ and $\beta_1 \subseteq \hat{U}$, let $d_b(\widehat{\ell_1}, \beta_1) = \inf\{d_b(\widehat{\ell_1}, \widehat{\ell_2}) : \widehat{\ell_2} \in \beta_1\}$. Denote $N(\hat{U})$, $CL(\hat{U})$, $CB(\hat{U})$ by the class all nonempty subsets of $\hat{U}$, the class of all nonempty closed subsets of $\hat{U}$ and the class of all nonempty closed and bounded subsets of $\hat{U}$ respectively. Define the Hausdorff-Pompeiu metric $\hat{H}_b$ induced by d_b on $CB(\hat{U})$ as follows:

$$\hat{H}_b(\beta_1, \beta_2) = \max\left\{\sup_{\widehat{\ell_1}\in\beta_1} d_b(\widehat{\ell_1}, \beta_2), \sup_{\widehat{\ell_2}\in\beta_2} d_b(\widehat{\ell_2}, \beta_1)\right\}$$

for all $\beta_1, \beta_2 \in CL(\hat{U})$. A point $\widehat{\ell} \in \hat{U}$ is said to be a fixed point of $\widetilde{T} : \hat{U} \to CL(\hat{U})$, if $\widehat{\ell} \in \widetilde{T}\widehat{\ell}$. If, for $\widehat{\ell_0} \in \hat{U}$, there exists a sequence $\{\widehat{\ell_i}\}$ in $\hat{U}$ such that $\widehat{\ell_i} \in \widetilde{T}\widehat{\ell_{i-1}}$, then $O(\widetilde{T}, \widehat{\ell_0}) = \{\widehat{\ell_0}, \widehat{\ell_1}, \widehat{\ell_2}, ...\}$ is said to be an orbit of $\widetilde{T} : \hat{U} \to CL(\hat{U})$. A mapping $f : \hat{U} \to \mathbb{R}$ is said to be $\widetilde{T}$-orbitally lower semi-continuous (o.l.s.c) if $\{\widehat{\ell_i}\}$ is a sequence in $O(\widetilde{T}, \widehat{\ell_0})$ and $\widehat{\ell_i} \to \varrho$ implies $f(\varrho) \leq \liminf_i f(\widehat{\ell_i})$.

From now on, Nadler [13] realized the following multivalued version of BCP:

Theorem 1.1. *[13] Let $(\hat{U}, d_b)$ be a complete metric space and $T : \hat{U} \to CB(\hat{U})$ be a Nadler contraction, i.e., there is $\gamma \in [0, 1)$ such that*

$$\hat{H}_b(T\widehat{\ell_1}, T\widehat{\ell_2}) \leq \gamma d_b(\widehat{\ell_1}, \widehat{\ell_2}) \text{ for all } \widehat{\ell_1}, \widehat{\ell_2} \in \hat{U}.$$

Then T possesses at least one fixed point.

We start the following results for main sequel.

Lemma 1.2. *[13] Let $(\hat{U}, d_b)$ be a metric space, $\beta_2 \in CB(\hat{U})$ and $\widehat{\ell} \in \hat{U}$. Then, for each $\epsilon > 0$, there exists $\nu \in \beta_2$ such that*

$$d_b(\widehat{\ell}, \nu) \leq d_b(\widehat{\ell}, \beta_2) + \epsilon.$$

Lemma 1.3. *[19] Let $(\hat{U}, d_b)$ be a metric space and $\beta_1, \beta_2 \in CB(\hat{U})$ with $\hat{H}_b(\beta_1, \beta_2) > 0$. Then for all $h > 1$ and $\widehat{\ell} \in \beta_1$, there exists $\nu = \nu(\widehat{\ell}) \in \beta_2$ such that*

$$d_b(\widehat{\ell}, \nu) < h\hat{H}_b(\beta_1, \beta_2).$$

There after, many researchers worked on existence of fixed point theorems of single valued mappings can improve in the module of multi-valued mappings that satisfying various classes of contractive mappings (see [1–4, 6, 9, 10, 12, 15, 17–20]).

Definition 1.4. [8] A b-metric space on a nonempty set M is a function $b : \hat{U} \times \hat{U} \to \mathbb{R}^+$ such that for all $\widehat{\ell_1}, \widehat{\ell_2}, \widehat{\ell_3} \in \hat{U}$ and a given real number $s \geq 1$, the following conditions hold:
(b_i) $d_b\left(\widehat{\ell_1}, \widehat{\ell_2}\right) = 0$ if and only if $\widehat{\ell_1} = \widehat{\ell_2}$;
(b_{ii}) $d_b\left(\widehat{\ell_1}, \widehat{\ell_2}\right) = d_b\left(\widehat{\ell_2}, \widehat{\ell_1}\right)$;
(b_{iii}) $d_b\left(\widehat{\ell_1}, \widehat{\ell_3}\right) \leq s[d_b\left(\widehat{\ell_1}, \widehat{\ell_2}\right) + d_b\left(\widehat{\ell_2}, \widehat{\ell_3}\right)]$.

The pair $\left(\hat{U}, d_b\right)$ is known as b-metric space.

The following examples present the context of b-metric spaces, which are essentially larger than the context of metric spaces [8].

Example 1.5. [8] Let $\hat{U} = l_p(\mathbb{R})$ with $p \in (0, 1)$ where $l_p(\mathbb{R}) = \{\left\{\widehat{\ell_i}\right\} \subset \mathbb{R} : \sum_{i=1}^{+\infty} \left|\widehat{\ell_i}\right|^p < \infty\}$. A function $b : \hat{U} \times \hat{U} \to \mathbb{R}^+$ is given by $b\left(\widehat{\ell_1}, \widehat{\ell_2}\right) = (\sum_{i=1}^{+\infty} \left|\widehat{\ell_i}\right|^p)^{\frac{1}{p}}$, where $\widehat{\ell_1} = \widehat{\ell_i}$ and $\widehat{\ell_2} = \widehat{\ell_{i'}}$. Then the pair $\left(\hat{U}, d_b\right)$ is known as b-metric space with $s = 2^{\frac{1}{p}}$.

Example 1.6. [8] Let $\hat{U} = L_p[0, 1]$ be the space of all real valued functions $\widehat{\ell}(r)$, $0 \leq r \leq 1$ in such a way that $\int_0^1 \left|\widehat{\ell}(r)\right|^{\frac{1}{p}} dr < \infty$. A function $b : \hat{U} \times \hat{U} \to \mathbb{R}^+$ is given by $b\left(\widehat{\ell_1}, \widehat{\ell_2}\right) = (\int_0^1 \left|\widehat{\ell_1}(r) - \widehat{\ell_2}(r)\right|^p)^{\frac{1}{p}}$. Then the pair $\left(\hat{U}, d_b\right)$ is known as b-metric space with $s = 2^{\frac{1}{p}}$.

Definition 1.7. [8] A sequence $\left\{\widehat{\ell_i}\right\}$ in b-metric space $\hat{U}$ is said to be convergent if there is $\widehat{\ell} \in \hat{U}$ such that $d_b\left(\widehat{\ell_i}, \widehat{\ell}\right) \to 0$ as $i \to +\infty$ and write $\lim_{i\to+\infty}(\widehat{\ell_i}) = \widehat{\ell}$. A sequence $\left\{\widehat{\ell_i}\right\}$ in $\left(\hat{U}, d_b\right)$ is said to be Cauchy if $d_b\left(\widehat{\ell_i}, \widehat{\ell_{i'}}\right) \to 0$ as $i, i' \to +\infty$. A b-metric space $\left(\hat{U}, d_b\right)$ is said to be complete if every Cauchy sequence in $\hat{U}$ converges.

Note that, in general, the b-metric is not a continuous functional. Recently, Liu *et al.* [12] produced the following classical function:

Definition 1.8. Let $\varphi : (0, +\infty) \to (0, +\infty)$ satisfy the following conditions:
(φ_a) φ is nondecreasing;
(φ_b) for all $\{\widehat{\ell_i}\}$ in $(0, +\infty)$, $\lim_{i\to+\infty} \varphi(\widehat{\ell_i}) = 0$ if and only if $\lim_{i\to+\infty}(\widehat{\ell_i}) = 0$;
(φ_c) φ is continuous.

From now on, we denote by φ^* the set of all function that satisfying $(\varphi_a) - (\varphi_c)$. The following well known two lammas of φ functions will be needed in our forthcoming sequel:

Lemma 1.9. *[12] Let $\left\{\widehat{\ell_i}\right\}_i$ be a bounded sequence of real numbers and all its convergent subsequences have the same limit γ. Then $\left\{\widehat{\ell_i}\right\}_i$ is convergent and* $\lim_{i\to+\infty}(\widehat{\ell_i}) = \gamma$.

Lemma 1.10. *Let $\varphi : (0, +\infty) \to (0, +\infty)$ be a nondecreasing and continuous function with* $\inf_{\widehat{\ell}\in(0,+\infty)} \varphi(\widehat{\ell}) = 0$ *and* $\left\{\widehat{\ell_i}\right\}_i \in (0, +\infty)$. *Then*

$$\lim_{i\to+\infty} \varphi(\widehat{\ell_i}) = 0 \textit{ if and only if } \lim_{i\to+\infty}(\widehat{\ell_i}) = 0.$$

Proof. ($\Rightarrow$) Suppose $\lim_{i\to+\infty} \varphi(\widehat{\ell_i}) = 0$. Then we claim that the sequence $\left\{\widehat{\ell_i}\right\}$ is bounded. In fact, if the sequence is unbounded, then we may assume that $\widehat{\ell_i} \to +\infty$ and so for all $\delta > 0$, there is $i_0 \in N$ such that $\widehat{\ell_i} > \delta$ for all $i > i_0$. Hence $\varphi(\delta) \leq \varphi(\widehat{\ell_i})$ and so $\varphi(\delta) \leq \lim_{i\to+\infty} \varphi(\widehat{\ell_i}) = 0$, which contradicts to $\varphi(\delta) > 0$. Thus $\left\{\widehat{\ell_i}\right\}$ is bounded. Hence there exists a subsequence $\left\{\widehat{\ell_{i_i}}\right\} \subset \left\{\widehat{\ell_i}\right\}$ such that $\lim_{i\to+\infty}\left\{\widehat{\ell_{i_i}}\right\} = k$ (where k is nonnegative number). Clearly $k \geq 0$. If $k > 0$, then there is $i_0 \in N$ such that $\left\{\widehat{\ell_{i_i}}\right\} \in \left(\frac{k}{2}, \frac{3k}{2}\right)$ for all $i \geq i_0$. By (φ_a), we deduce that $\varphi(\frac{k}{2}) \leq \lim_{i\to+\infty}\left\{\widehat{\ell_{i_i}}\right\} = 0$, which contradicts to $\varphi(\frac{k}{2}) > 0$. Consequently, setting $k = 0$ and by the above lemma, we have $\lim_{i\to+\infty}(\widehat{\ell_i}) = 0$.

($\Leftarrow$) Suppose that $inf_{\widehat{\ell}\in(0,+\infty)}\varphi(\widehat{\ell}) = 0$. If $\widehat{\ell_i} \to 0$, then for any given $\epsilon > 0$, there is $k > 0$ such that $\varphi(k) \in (0, \epsilon)$ and there exists $i_1 \in N$ such that $\widehat{\ell_i} < k$ for all $i > i_1$. Therefore, $0 < \varphi(\widehat{\ell_i}) \leq \varphi(k) < \epsilon$ for $i > i_1$. Hence $\varphi(\widehat{\ell_i}) \to 0$ as $i \to +\infty$. □

Throughout this paper E denotes an interval on $\mathbb{R}^+$ containing 0, that is, an interval of the form $[0, R]$, $[0, R)$, or $[0, +\infty)$. Proinov [14] introduced the following:

Lemma 1.11. *[14] Let $\widehat{\ell_0} \in \Lambda$ (Λ is a closed subset of $\hat{U}$) such that*

$$d_b(\widehat{\ell_0}, \widehat{T\ell_0}) \in E,$$

and $\widehat{\ell_i} \in \Lambda$ for some $i \geq 0$. Then we have $d_b(\widehat{\ell_i}, \widehat{T\ell_i}) \in E$.

Definition 1.12. [14] Suppose $\widehat{\ell_0} \in \Lambda$ and $d_b(\widehat{\ell_0}, \widehat{T\ell_0}) \in E$. Then for an iterate $\widehat{\ell_i}$ ($i \geq 0$) which belongs to Λ, we define the closed ball $\overline{b}(\widehat{\ell_i}, \rho)$ with center $\widehat{\ell_i}$ and radius $\rho > 0$.

Lemma 1.13. *[14] If an element* $\widehat{\ell_0} \in \Lambda$ *satisfies* $d_b(\widehat{\ell_0}, \widetilde{T\ell_0}) \in E$ *and* $\overline{b}(\widehat{\ell_i}, \rho) \subset \Lambda$ *for some* $i \geq 0$, *then* $\widehat{\ell_{i+1}} \in \Lambda$ *and* $\overline{b}(\widehat{\ell_{i+1}}, \rho) \subset \overline{b}(\widehat{\ell_i}, \rho)$.

Definition 1.14. [14] Let $i \geq 1$. A function $\xi : E \to E$ is said to be a gauge function of order i on E if it satisfies the following conditions: (a) $\xi(\lambda\widehat{\ell}) < \lambda^i \xi(\widehat{\ell})$ for all $\lambda \in (0, 1)$ and $\widehat{\ell} \in E$; (b) $\xi(\widehat{\ell}) < \widehat{\ell}$ for all $\widehat{\ell} \in E - \{0\}$.

It is easy to see that the first condition of Definition 1.14 is equivalent to the following: $\xi(0) = 0$ and $\xi(\widehat{\ell})/\widehat{\ell}^i$ is nondecreasing on $E - \{0\}$.

Definition 1.15. [14] A gauge function $\xi : E \to E$ is said to be a B-GGF on E if

$$\sigma(\widehat{\ell}) = \sum_{i=0}^{+\infty} \xi^i(\widehat{\ell}) < \infty, \quad \text{for all } \widehat{\ell} \in E.$$

Note that a B-GGF also satisfies the following functional equation:

$$\sigma(\widehat{\ell}) = \sigma(\xi(\widehat{\ell})) + \widehat{\ell}.$$

Proinov [14] proved his main results by assuming B-GGF ξ and the mapping $T : \Lambda \to X$ satisfying the contractive condition $d(T(x) T^2(x)) \leq \xi(d(x; Tx))$ when the underlying space is endowed with a metric. But from now on, in the context of b-metric space for some technical dialectics, Samreen *et al.* [16] introduced the following class of GF.

Definition 1.16. [16] A nondecreasing function $\xi : E \to E$ is said to be a b-B-GGF on E if

$$\sigma(\widehat{\ell}) = \sum_{i=0}^{+\infty} s^i \xi^i(\widehat{\ell}) < \infty, \quad \text{for all } \widehat{\ell} \in E$$

where s is the coefficient of b-metric space. Moreover, note that a b-B-GGF also satisfies the following functional equation:

$$\sigma(\widehat{\ell}) = s\sigma(\xi(\widehat{\ell})) + \widehat{\ell}.$$

Remark 1.17. Every b-B-GGF is also a B-GGF [7] but the converse may not hold. Furthermore, in [16], Samreen *et al.* introduced gauge functions in a b-metric space of the form

$$\xi(\widehat{\ell}) = \begin{cases} \frac{s\xi(\widehat{\ell})}{\widehat{\ell}}, \text{ if } \widehat{\ell} \in E - \{0\} \\ 0, \text{ if } \widehat{\ell} = 0 \end{cases}$$

where s is the coefficient of b-metric space. For instance, we refer the following simple examples of gauge functions of order i as:

(a) $\xi(\widehat{\ell}) = \frac{\lambda\widehat{\ell}}{s}$ for all $\lambda \in (0, 1)$ is a gauge function of order 1 on $\widehat{\ell} \in E$;

(b) $\xi(\widehat{\ell}) = \frac{\lambda\widehat{\ell}^k}{s}$ $(\lambda > 0, k > 0)$ is a gauge function of order k on $E = [0, l)$ where $l = \left(\frac{1}{\lambda}\right)^{\frac{1}{1-k}}$.

In 2015, Khojasteh *et al.* [11] introduced the concept of simulation function as follows:

Definition 1.18. [11] A function $\Gamma : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called an SF if
(Γ1) $\Gamma(0,0) = 0$;
(Γ2) $\Gamma(\widehat{\ell_1}, \widehat{\ell_2}) < \widehat{\ell_2} - \widehat{\ell_1}$ for all $\widehat{\ell_1}, \widehat{\ell_2} > 0$;
(Γ3) if $\{\widehat{\ell_{1i}}\}$,$\{\widehat{\ell_{2i}}\} \in (0, +\infty)$ such that $\lim_{i\to+\infty} \widehat{\ell_{1i}} = \lim_{i\to+\infty} \widehat{\ell_{2i}} > 0$, then

$$\limsup_{i\to+\infty} \Gamma(\widehat{\ell_{1i}}, \widehat{\ell_{2i}}) < 0.$$

Due to (Γ2), we have $\Gamma(\widehat{\ell_1}, \widehat{\ell_1}) < 0$ for all $\widehat{\ell_1} > 0$. From now on, we denote by ∇ the set of all functions satisfying (Γ1)-(Γ3). Some well known examples of Γ functions presented in the existing exposition are as follows:

Example 1.19. [11] For $i = 1, 2$, let $\vartheta_i : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous functions with $\vartheta_i(\widehat{\ell_1}) = 0$ if and only if $\widehat{\ell_1} = 0$. The following functions $\Gamma_j : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ ($j = 1, \cdots, 6$) are in ∇:

(a) $\Gamma_1(\widehat{\ell_1}, \widehat{\ell_2}) = \vartheta_1(\widehat{\ell_2}) - \vartheta_2(\widehat{\ell_1})$ for all $\widehat{\ell_1}, \widehat{\ell_2} \geq 0$, where $\vartheta_1(\widehat{\ell_1}) \leq \widehat{\ell_1} \leq \vartheta_2(\widehat{\ell_1})$ for all $\widehat{\ell_1} > 0$;
(b) $\Gamma_6(\widehat{\ell_1}, \widehat{\ell_2}) = \widehat{\ell_2} - \int_0^{\widehat{\ell_1}} \varsigma(u)du$ for all $\widehat{\ell_1}, \widehat{\ell_2} \geq 0$, where $\varsigma : \mathbb{R}^+ \to \mathbb{R}^+$ is a function such that

$$\int_0^{\epsilon} \varsigma(u)du \text{ exists and } \int_0^{\epsilon} \varsigma(u)du > \epsilon \;\; \forall \epsilon > 0.$$

Let $(\hat{U}, d_b)$ be a metric space, $\widetilde{T}$ be a self mapping on $\hat{U}$ and $\Gamma \in \nabla$. $\widetilde{T}$ is said to be a ∇-contraction with respect to Γ, if

$$\Gamma(d_b(\widetilde{T\widehat{\ell_1}}, \widetilde{T\widehat{\ell_2}}), d_b(\widehat{\ell_1}, \widehat{\ell_2})) \geq 0, \quad \text{for all } \widehat{\ell_1}, \widehat{\ell_2} \in \hat{U}.$$

Due to (Γ_2), we have $d_b(T\widehat{\ell_1}, T\widehat{\ell_2}) \neq d_b(\widehat{\ell_1}, \widehat{\ell_2})$ for all distinct points $\widehat{\ell_1}, \widehat{\ell_2} \in \hat{U}$. Thus T is not an isometry, whenever T is a ∇-contraction with respect to Γ. Conversely, if a ∇-contraction mapping T on a metric space possesses a fixed point, then it is necessarily unique.

In the recent year, Ali *et al.* [5] initiated the following definition which is a modification of the notion of α-admissible.

Definition 1.20. [5] Let $(\hat{U}, d_b)$ be a metric space and Λ be a nonempty subset of $\hat{U}$. A mapping $\widetilde{T} : \Lambda \to CB(\hat{U})$ is called α-admissible if there exists a function $\alpha : \Lambda \times \Lambda \to [0, +\infty)$ such that

$$\alpha(a, b) \geq 1 \quad \Rightarrow \quad \alpha(\widehat{\ell}, \nu) \geq 1,$$

for all $\widehat{\ell} \in \widetilde{T}a \cap \Lambda$ and $\nu \in \widetilde{T}b \cap \Lambda$.

In this manuscript, we prove the notion of multi-valued Suzuki (SU) type fixed point results via φ_ξ-contraction mapping and $(\nabla_\alpha - \xi)$-contraction mapping in the module of b-metric spaces, where ξ is a b-B-GGF on an interval E with some tangible examples and certain important corollaries are adopted subsequently. Our newly proved results over recent ones chiefly due to Proinov [14] and Ali *et al.* [1]. As the end results of a succession, we promote our main results to prove the existence of solution for the system of integral inclusion.

2. Multivalued SU-type φ_ξ-contraction

In this section, motivated by the notion of multivalued Suzuki type φ-contraction, we define the notion of multivalued Suzuki type φ_ξ-contraction as follows:

Definition 2.1. Let $(\hat{U}, d_b)$ be a b-metric space with $s \geq 1$, Λ be a closed subset of $\hat{U}$ and ξ be a b-B-GGF on an interval E. A mapping $\widetilde{T} : \Lambda \to CB(\hat{U})$ is said to be a multivalued SU-type φ-contraction if there exists $\varphi \in \varphi^*$ such that for $\widetilde{T\ell} \cap \Lambda \neq \emptyset$

$$\frac{1}{2s}\min\left\{d_b(\widehat{\ell}, \widetilde{T\ell} \cap \Lambda), d_b(\nu, \widetilde{T}\nu \cap \Lambda)\right\} < d_b(\widehat{\ell}, \nu)$$

implies that

$$\varphi\left[\hat{H}_b(\widetilde{T\ell} \cap \Lambda, \widetilde{T}\nu \cap \Lambda)\right] \leq \varphi\left[\xi(\Omega(\widehat{\ell}, \nu))\right], \tag{2.1}$$

where

$$\Omega(\widehat{\ell}, \nu) = \max\left\{d_b(\widehat{\ell}, \nu), d_b(\widehat{\ell}, \widetilde{T\ell}), d_b(\nu, \widetilde{T}\nu), \frac{d_b(\widehat{\ell}, \widetilde{T}\nu) + d_b(\nu, \widetilde{T\ell})}{2s}\right\}$$

for all $\widehat{\ell} \in \Lambda$, $\nu \in \widetilde{T\ell} \cap \Lambda$ with $d_b(\widehat{\ell}, \nu) \in E$, and $\hat{H}_b(\widetilde{T\ell} \cap \Lambda, \widetilde{T}\nu \cap \Lambda) > 0$.

Clearly in a class b-metric space, if an element $\widehat{\ell_0} \in \Lambda$ such that $O(\widehat{\ell_0}) \subset \Lambda$ satisfies $d_b(\widehat{\ell_0}, \widetilde{T\ell_0}) \in E$ and $\overline{b}(\widehat{\ell_i}, \rho_i) \subset \Lambda$ for some $i \geq 0$, then $\widehat{\ell_{i+1}} \in \Lambda$ and $\overline{b}(\widehat{\ell_{i+1}}, \rho_{i+1}) \subset \overline{b}(\widehat{\ell_i}, \rho_i)$.

Our first main result is as follows:

Theorem 2.2. *Let $(\hat{U}, d_b)$ be a complete b-metric space with $s \geq 1$, Λ be a closed subset of $\hat{U}$ and $\widetilde{T} : \Lambda \to CB(\hat{U})$ be a multivalued SU-type φ-contraction. Assume $\widehat{\ell_0} \in \Lambda$ such that $d_b(\widehat{\ell_0}, c^*) \in E$ for some $c^* \in \widetilde{T\ell_0} \cap \Lambda$. Then there exist an orbit $\{\widehat{\ell_i}\}$ of $\widetilde{T}$ in Λ and $\sigma^* \in \Lambda$ such that $\lim_{i\to+\infty}\widehat{\ell_i} = \sigma^*$. Moreover, σ^* is a fixed point of $\widetilde{T}$ if and only if the function $g(\widehat{\ell}) := d_b(\widehat{\ell}, \widetilde{T\ell} \cap \Lambda)$ is $\widetilde{T}$-o.l.s.c at σ^*.*

Proof. Choose $\widehat{\ell_1} = c^* \in \widetilde{T\ell_0} \cap \Lambda$. In the presence of this manner $d_b(\widehat{\ell_0}, \widehat{\ell_1}) = 0$, $\widehat{\ell_0}$ is a fixed point of $\widetilde{T}$. Thus we assume that $d_b(\widehat{\ell_0}, \widehat{\ell_1}) \neq 0$. On the other hand, we have

$$\frac{1}{2s}\min\left\{d_b(\widehat{\ell_0}, \widetilde{T\ell_0} \cap \Lambda), d_b(\widehat{\ell_1}, \widetilde{T\ell_1} \cap \Lambda)\right\} < d_b(\widehat{\ell_0}, \widehat{\ell_1}). \tag{2.2}$$

Define $\rho = \sigma(d_b(\widehat{\ell_0}, \widehat{\ell_1}))$. From (1.16), we have $\sigma(r) \geq r$. Hence $d_b(\widehat{\ell_0}, \widehat{\ell_1}) \leq \rho$ and so $\widehat{\ell_1} \in \overline{b}(\widehat{\ell_0}, \rho)$. Since $d_b(\widehat{\ell_0}, \widehat{\ell_1}) \in E$, from (2.1) and (2.2) it follows that

$$\varphi\left[H_b(\widetilde{T\ell_0} \cap \Lambda, \widetilde{T\ell_1} \cap \Lambda)\right] \leq \varphi\left[\xi(\Omega(\widehat{\ell_0}, \widehat{\ell_1}))\right] < \varphi\left[\Omega(\widehat{\ell_0}, \widehat{\ell_1})\right].$$

By the property of right continuity of φ, there exists a real number $h_1 > 1$ such that

$$\varphi\left[h_1 H_b(\widetilde{T\ell_0} \cap \Lambda, \widetilde{T\ell_1} \cap \Lambda)\right] \leq \varphi\left[\xi(\Omega(\widehat{\ell_0}, \widehat{\ell_1}))\right]. \tag{2.3}$$

From

$$d_b(\widehat{\ell_1}, \widetilde{T\ell_1} \cap \Lambda) \leq H_b(\widetilde{T\ell_0} \cap \Lambda, \widetilde{T\ell_1} \cap \Lambda) < h_1 H_b(\widetilde{T\ell_0} \cap \Lambda, \widetilde{T\ell_1} \cap \Lambda),$$

by Lemma 1.3, there exists $\widehat{\ell_2} \in \widetilde{T\ell_1} \cap \Lambda$ such that $d_b(\widehat{\ell_1}, \widehat{\ell_2}) \leq h_1 H_b(\widetilde{T\ell_0} \cap \Lambda, \widetilde{T\ell_1} \cap \Lambda)$. Since φ is nondecreasing, by (2.3), this inequality gives that

$$\varphi\left[(d_b(\widehat{\ell_1}, \widehat{\ell_2})\right] \leq \varphi[h_1 H_b(\widetilde{T\ell_0} \cap \Lambda, \widetilde{T\ell_1} \cap \Lambda)] < \varphi\left[\Omega(\widehat{\ell_0}, \widehat{\ell_1}))\right],$$

where

$$\begin{aligned}\Omega(\widehat{\ell_0},\widehat{\ell_1}) &= \max\left\{d_b(\widehat{\ell_0},\widehat{\ell_1}), d_b(\widehat{\ell_0},\widetilde{T\ell_0}), d_b(\widehat{\ell_1},\widetilde{T\ell_1}), \frac{d_b(\widehat{\ell_0},\widetilde{T\ell_1})+d_b(\widehat{\ell_1},\widetilde{T\ell_0})}{2s}\right\}\\ &\le \max\left\{d_b(\widehat{\ell_0},\widehat{\ell_1}), d_b(\widehat{\ell_1},\widetilde{T\ell_1}), \frac{d_b(\widehat{\ell_0},\widetilde{T\ell_1})}{2s}\right\}\\ &\le \max\left\{d_b(\widehat{\ell_0},\widehat{\ell_1}), d_b(\widehat{\ell_1},\widetilde{T\ell_1})\right\}.\end{aligned}$$

Now, we claim that

$$\varphi\left[(d_b(\widehat{\ell_1},\widehat{\ell_2})\right] \le \varphi[h_1 H_b(\widetilde{T\ell_0}\cap\Lambda, \widetilde{T\ell_1}\cap\Lambda)] < \varphi\left[d_b(\widehat{\ell_0},\widehat{\ell_1}))\right]. \tag{2.4}$$

Let $\Delta = \max\left\{d_b(\widehat{\ell_0},\widehat{\ell_1}), d_b(\widehat{\ell_1},\widetilde{T\ell_1})\right\}$. Assume that $\Delta = d_b(\widehat{\ell_1},\widetilde{T\ell_1})$. Since $\widehat{\ell_2}\in\widetilde{T\ell_1}\cap\Lambda$, we have

$$\varphi\left[(d_b(\widehat{\ell_1},\widehat{\ell_2})\right] \le \varphi[h_1 H_b(\widetilde{T\ell_0}\cap\Lambda, \widetilde{T\ell_1}\cap\Lambda)] < \varphi\left[d_b(\widehat{\ell_1},\widehat{\ell_2}))\right],$$

which is a contradiction. Hence (2.4) holds true. We assume that $d_b(\widehat{\ell_1},\widehat{\ell_2})\neq 0$, otherwise, $\widehat{\ell_1}$ is a fixed point of $\widetilde{T}$. From (φ_a), (2.4) implies that

$$d_b(\widehat{\ell_1},\widehat{\ell_2}) < d_b(\widehat{\ell_0},\widehat{\ell_1}).$$

and so $d_b(\widehat{\ell_1},\widehat{\ell_2})\in E$. Next, $\widehat{\ell_2}\in\overline{b}(\widehat{\ell_0},\rho)$ since

$$\begin{aligned}d_b(\widehat{\ell_0},\widehat{\ell_2}) \le s d_b(\widehat{\ell_0},\widehat{\ell_1}) + s d_b(\widehat{\ell_1},\widehat{\ell_2}) &\le s d_b(\widehat{\ell_0},\widehat{\ell_1}) + s^2 d_b(\widehat{\ell_1},\widehat{\ell_2})\\ &\le s d_b(\widehat{\ell_0},\widehat{\ell_1}) + s^2\xi(d_b(\widehat{\ell_0},\widehat{\ell_1}))\\ &= s\left[d_b(\widehat{\ell_0},\widehat{\ell_1}) + s\xi(d_b(\widehat{\ell_0},\widehat{\ell_1}))\right]\\ &\le s\sigma d_b(\widehat{\ell_0},\widehat{\ell_1})\\ &\le d_b(\widehat{\ell_0},\widehat{\ell_1}) + s\sigma(d_b(\widehat{\ell_0},\widehat{\ell_1}))\\ &= \sigma(d_b(\widehat{\ell_0},\widehat{\ell_1})) = \rho.\end{aligned}$$

Since

$$\frac{1}{2s}\min\left\{d_b(\widehat{\ell_1},\widetilde{T\ell_1}\cap\Lambda), d_b(\widehat{\ell_2},\widetilde{T\ell_2}\cap\Lambda)\right\} < d_b(\widehat{\ell_1},\widehat{\ell_2}),$$

from (2.1), we have

$$\varphi\left[H_b(\widetilde{T\ell_1}\cap\Lambda, \widetilde{T\ell_2}\cap\Lambda)\right] \le \varphi\left[\xi(d_b(\widehat{\ell_1},\widehat{\ell_2})))\right] < \varphi\left[\Omega(\widehat{\ell_1},\widehat{\ell_2}))\right].$$

Since φ is right continuous, there exists a real number $h_2 > 1$ such that

$$\varphi\left[h_2 H_b(\widetilde{T\ell_1}\cap\Lambda, \widetilde{T\ell_2}\cap\Lambda\right] \le \varphi\left[\xi(\Omega(\widehat{\ell_1},\widehat{\ell_2}))\right]. \tag{2.5}$$

Next, from

$$d_b(\widehat{\ell_2},\widetilde{T\ell_2}\cap\Lambda) \le H_b(\widetilde{T\ell_1}\cap\Lambda, \widetilde{T\ell_2}\cap\Lambda) < h_2 H_b(\widetilde{T\ell_1}\cap\Lambda, \widetilde{T\ell_2}\cap\Lambda),$$

by Lemma 1.3, there exists $\widehat{\ell_3} \in \widetilde{T\widehat{\ell_2}} \cap \Lambda$ such that $d_b(\widehat{\ell_2}, \widehat{\ell_3}) \leq h_2 H_b(\widetilde{T\widehat{\ell_1}} \cap \Lambda, \widetilde{T\widehat{\ell_2}} \cap \Lambda)$. By (2.5), this inequality gives that

$$\varphi\left[(d_b(\widehat{\ell_2}, \widehat{\ell_3}))\right] \leq \varphi\left[h_2 H_b(\widetilde{T\widehat{\ell_1}} \cap \Lambda, \widetilde{T\widehat{\ell_2}} \cap \Lambda)\right] < \varphi\left[\Omega(\widehat{\ell_1}, \widehat{\ell_2}))\right],$$

where

$$\begin{aligned} \Omega(\widehat{\ell_1}, \widehat{\ell_2}) &= \max\left\{d_b(\widehat{\ell_1}, \widehat{\ell_2}), d_b(\widehat{\ell_1}, \widetilde{T\widehat{\ell_1}}), d_b(\widehat{\ell_2}, \widetilde{T\widehat{\ell_2}}), \frac{d_b(\widehat{\ell_1}, \widetilde{T\widehat{\ell_2}}) + d_b(\widehat{\ell_2}, \widetilde{T\widehat{\ell_1}})}{2s}\right\} \\ &\leq \max\left\{\widehat{d(\widehat{\ell_1}, \widehat{\ell_2})}, \widehat{d(\widehat{\ell_2}, \widetilde{T\widehat{\ell_2}})}, \frac{\widehat{d(\widehat{\ell_1}, \widetilde{T\widehat{\ell_2}})}}{2s}\right\} \\ &\leq \max\left\{\widehat{d(\widehat{\ell_1}, \widehat{\ell_2})}, \widehat{d(\widehat{\ell_2}, \widetilde{T\widehat{\ell_2}})}\right\}. \end{aligned}$$

This implies that

$$\varphi\left[(\widehat{d(\widehat{\ell_2}, \widehat{\ell_3})}\right] \leq \varphi[h_1 H_b(\widetilde{T\widehat{\ell_1}} \cap \Lambda, \widetilde{T\widehat{\ell_2}} \cap \Lambda)] < \varphi\left[\widehat{d(\widehat{\ell_1}, \widehat{\ell_2})})\right]. \tag{2.6}$$

Let $\Delta = \max\left\{d_b(\widehat{\ell_1}, \widehat{\ell_2}), d_b(\widehat{\ell_2}, \widetilde{T\widehat{\ell_2}})\right\}$. Assume that $\Delta = d_b(\widehat{\ell_2}, \widetilde{T\widehat{\ell_2}})$. Since $\widehat{\ell_3} \in \widetilde{T\widehat{\ell_2}} \cap \Lambda$, we have

$$\varphi\left[(d_b(\widehat{\ell_2}, \widehat{\ell_3})\right] \leq \varphi[h_1 H_b(\widetilde{T\widehat{\ell_1}} \cap \Lambda, \widetilde{T\widehat{\ell_2}} \cap \Lambda)] < \varphi\left[d_b(\widehat{\ell_2}, \widehat{\ell_3}))\right],$$

which is a contradiction. Hence (2.6) holds true. We assume that $d_b(\widehat{\ell_2}, \widehat{\ell_3}) \neq 0$, otherwise, $\widehat{\ell_2}$ is a fixed point of $\widetilde{T}$. From (φ_a), (2.6) implies that

$$d_b(\widehat{\ell_2}, \widehat{\ell_3}) < d_b(\widehat{\ell_1}, \widehat{\ell_2}).$$

and so $d_b(\widehat{\ell_2}, \widehat{\ell_3}) \in E$. Also, we have $\widehat{\ell_3} \in \overline{b}(\widehat{\ell_0}, \rho)$, since

$$\begin{aligned} d_b(\widehat{\ell_0}, \widehat{\ell_3}) \leq s d_b(\widehat{\ell_0}, \widehat{\ell_1}) + s^2 d_b(\widehat{\ell_1}, \widehat{\ell_2}) + s^3 d_b(\widehat{\ell_2}, \widehat{\ell_3}) &= s\left[d_b(\widehat{\ell_0}, \widehat{\ell_1}) + s\check{d}_b(\widehat{\ell_1}, \widehat{\ell_2}) + s^2 d_b(\widehat{\ell_2}, \widehat{\ell_3})\right] \\ &\leq s\left[d_b(\widehat{\ell_0}, \widehat{\ell_1}) + \xi(d_b(\widehat{\ell_0}, \widehat{\ell_1})) + \xi^2(d_b(\widehat{\ell_0}, \widehat{\ell_1}))\right] \\ &\leq s\sigma\check{d}_b(\widehat{\ell_0}, \widehat{\ell_1}) \\ &\leq d_b(\widehat{\ell_0}, \widehat{\ell_1}) + s\sigma(d_b(\widehat{\ell_0}, \widehat{\ell_1})) \\ &= \sigma(d_b(\widehat{\ell_0}, \widehat{\ell_1})) = \rho. \end{aligned}$$

Continuing this manner, we build two sequences $\left\{\widehat{\ell_i}\right\} \subset \overline{b}(\widehat{\ell_0}, \rho)$ and $\{h_i\} \subset (0, +\infty)$ such that $\widehat{\ell_{i+1}} \in \widetilde{T\widehat{\ell_i}} \cap \Lambda$, $\widehat{\ell_i} \neq \widehat{\ell_{i+1}}$ with $d_b(\widehat{\ell_i}, \widehat{\ell_{i+1}}) \in E$ and

$$\varphi\left[(d_b(\widehat{\ell_i}, \widehat{\ell_{i+1}}))\right] \leq \varphi\left[h_i H_b(\widetilde{T\widehat{\ell_{i-1}}} \cap \Lambda, \widetilde{T\widehat{\ell_i}} \cap \Lambda)\right] < \varphi\left[d_b(\widehat{\ell_{i-1}}, \widehat{\ell_i})\right],$$

for all $i \in \mathbb{N}$. Then

$$\varphi\left[d_b(\widehat{\ell_i}, \widehat{\ell_{i+1}})\right] \leq \varphi\left[\xi^i\left(\check{d}_b(\widehat{\ell_0}, \widehat{\ell_1})\right)\right], \text{ for all } i \in \mathbb{N}.$$

Since $\varphi : (0, +\infty) \to (0, +\infty)$, it follows from (2.6) that

$$0 \leq \lim_{i \to +\infty} \varphi\left[d_b(\widehat{\ell_i}, \widehat{\ell_{i+1}})\right] \leq \lim_{i \to +\infty} \varphi\left[\xi^i\left(d_b(\widehat{\ell_0}, \widehat{\ell_1})\right)\right] = 0,$$

which implies that

$$\lim_{i\to+\infty}\varphi\left[d_b(\widehat{\ell_i},\widehat{\ell_{i+1}})\right]=0.$$

By (φ_b) and Lemma 1.2, we have

$$\lim_{i\to+\infty}\check{d}_b(\widehat{\ell_i},\widehat{\ell_{i+1}})=0. \tag{2.7}$$

Next, we prove that $\{\widehat{\ell_i}\}$ is a Cauchy sequence in $\hat{U}$. Arguing by contradiction, we assume that there are $\epsilon>0$ and sequences $\{\delta_i\}_{i=1}^{+\infty}$ and $\{\kappa_i\}_{i=1}^{+\infty}$ of natural numbers such that

$$\delta_i>\kappa_i>0,\ d_b(\widehat{\ell_{\delta_i}},\widehat{\ell_{\kappa_i}})\geq\epsilon \text{ and } d_b(\widehat{\ell_{\delta_i-1}},\widehat{\ell_{\kappa_i}})<\epsilon \text{ for all } i\in\mathbb{N}.$$

Therefore,

$$\begin{aligned}\epsilon &\leq d_b(\widehat{\ell_{\delta_i}},\widehat{\ell_{\kappa_i}}) \\ &\leq s\left[d_b(\widehat{\ell_{\delta_i}},\widehat{\ell_{\delta_i-1}})+d_b(\widehat{\ell_{\delta_i-1}},\widehat{\ell_{\kappa_i}})\right] \\ &\leq s\check{d}_b(\widehat{\ell_{\delta_i}},\widehat{\ell_{\delta_i-1}})+s\epsilon.\end{aligned} \tag{2.8}$$

Setting $i\to+\infty$ in (2.8),

$$\epsilon<\lim_{i\to+\infty}d_b(\widehat{\ell_{\delta_i}},\widehat{\ell_{\kappa_i}})<s\epsilon. \tag{2.9}$$

From the trianguler inequality, we have

$$d_b(\widehat{\ell_{\delta_i}},\widehat{\ell_{\kappa_i}})\leq d_b(\widehat{\ell_{\delta_i}},\widehat{\ell_{\delta_i+1}})+d_b(\widehat{\ell_{\delta_i+1}},\widehat{\ell_{\kappa_i}}) \tag{2.10}$$

and

$$d_b(\widehat{\ell_{\delta_i+1}},\widehat{\ell_{\kappa_i}})\leq s\left[\check{d}_b(\widehat{\ell_{\delta_i}},\widehat{\ell_{\delta_i+1}})+d_b(\widehat{\ell_{\delta_i}},\widehat{\ell_{\kappa_i}})\right]. \tag{2.11}$$

Letting the upper limit as $i\to+\infty$ in (2.10) and applying (2.7) and (2.9), we obtain

$$\epsilon\leq\lim_{i\to+\infty}\sup d_b(\widehat{\ell_{\delta_i}},\widehat{\ell_{\kappa_i}})\leq s\left[\lim_{i\to+\infty}\sup d_b(\widehat{\ell_{\delta_i+1}},\widehat{\ell_{\kappa_i}})\right].$$

Again, setting the upper limit as $i\to+\infty$ in (2.11), we get

$$\lim_{i\to+\infty}\sup d_b(\widehat{\ell_{\delta_i+1}},\widehat{\ell_{\kappa_i}})\leq s\left[\lim_{i\to+\infty}\sup d_b(\widehat{\ell_{\delta_i}},\widehat{\ell_{\kappa_i}})\right]\leq s.s\epsilon=s^2\epsilon.$$

Therefore,

$$\frac{\epsilon}{s}\leq\lim_{i\to+\infty}\sup d_b(\widehat{\ell_{\delta_i+1}},\widehat{\ell_{\kappa_i}})\leq s^2\epsilon, \tag{2.12}$$

equivalently, we have

$$\frac{\epsilon}{s}\leq\lim_{i\to+\infty}\sup d_b(\widehat{\ell_{\delta_i}},\widehat{\ell_{\kappa_i+1}})\leq s^2\epsilon. \tag{2.13}$$

By the trianguler inequality,

$$\check{d}_b(\widehat{\ell_{\delta_i+1}},\widehat{\ell_{\kappa_i}})\leq s\left[d_b(\widehat{\ell_{\delta_i+1}},\widehat{\ell_{\kappa_i+1}})+d_b(\widehat{\ell_{\kappa_i+1}},\widehat{\ell_{\kappa_i}})\right]. \tag{2.14}$$

Setting the limit as $i\to+\infty$ in (2.14), using (2.7) and (2.12), we have

$$\frac{\epsilon}{s^2}\leq\lim_{i\to+\infty}\sup d_b(\widehat{\ell_{\delta_i+1}},\widehat{\ell_{\kappa_i+1}}). \tag{2.15}$$

Owing to above process, we find

$$\lim_{i\to+\infty}\sup \check{d}_b(\widehat{\ell_{\delta_i+1}},\widehat{\ell_{\kappa_i+1}})\leq s^3\epsilon. \tag{2.16}$$

From (2.15) and (2.16), we have

$$\frac{\epsilon}{s^2}\leq \lim_{i\to+\infty}\sup d_b(\widehat{\ell_{\delta_i+1}},\widehat{\ell_{\kappa_i+1}})\leq s^3\epsilon.$$

Owing to (2.7) and (2.9), we can choose a positive integer $j_0\geq 1$ such that

$$\frac{1}{2s}\min\left\{d_b(\widehat{\ell_{\delta_i}},\widetilde{T\widehat{\ell_{\delta_i}}}\cap\Lambda),d_b(\widehat{\ell_{\kappa_i}},\widetilde{T\widehat{\ell_{\kappa_i}}}\cap\Lambda)\right\}<\frac{\epsilon}{2s}<\check{d}_b(\widehat{\ell_{\delta_i}},\widehat{\ell_{\kappa_i}})$$

for all $i\geq j_0$. From (2.1), we have

$$0<\varphi\left[d_b(\widehat{\ell_{\delta_i+1}},\widehat{\ell_{\kappa_i+1}})\right]\leq\varphi\left[H_b(\widetilde{T\widehat{\ell_{\delta_i}}}\cap\Lambda,\widetilde{T\widehat{\ell_{\kappa_i}}}\cap\Lambda)\right]\leq\varphi\left[\xi(\Omega(\widehat{\ell_{\delta_i}},\widehat{\ell_{\kappa_i}})))\right],$$

where

$$\begin{aligned}\Omega(\widehat{\ell_{\delta_i}},\widehat{\ell_{\kappa_i}}) &= \max\left\{\begin{array}{c} d_b(\widehat{\ell_{\delta_i}},\widehat{\ell_{\kappa_i}}),d_b(\widehat{\ell_{\delta_i}},\widetilde{T\widehat{\ell_{\delta_i}}}),d_b(\widehat{\ell_{\kappa_i}},\widetilde{T\widehat{\ell_{\kappa_i}}}),\\ \frac{d_b(\widehat{\ell_{\delta_i}},\widetilde{T\widehat{\ell_{\kappa_i}}})+d_b(\widehat{\ell_{\kappa_i}},\widetilde{T\widehat{\ell_{\delta_i}}})}{2s}\end{array}\right\}\\ &\leq \max\left\{\begin{array}{c} d_b(\widehat{\ell_{\delta_i}},\widehat{\ell_{\kappa_i}}),d_b(\widehat{\ell_{\delta_i}},\widehat{\ell_{\delta_i+1}}),d_b(\widehat{\ell_{\kappa_i}},\widehat{\ell_{\kappa_i+1}}),\\ \frac{d_b(\widehat{\ell_{\delta_i}},\widehat{\ell_{\kappa_i+1}})+d_b(\widehat{\ell_{\kappa_i}},\widehat{\ell_{\delta_i+1}})}{2s}\end{array}\right\}.\end{aligned}$$

Setting the limit as $i\to+\infty$ and by (2.7), (2.9), (2.12) and (2.13), we have

$$\begin{aligned}\epsilon &= \max\left\{\epsilon,\frac{1}{2s}\left(\frac{\epsilon}{s}+\frac{\epsilon}{s}\right)\right\}\\ &\leq \lim_{i\to+\infty}\sup\Omega(\widehat{\ell_{\delta_i}},\widehat{\ell_{\kappa_i}})\\ &\leq \max\left\{s\epsilon,\frac{1}{2s}\left(s^2\epsilon+s^2\epsilon\right)\right\}=s\epsilon.\end{aligned}$$

By (2.15) and (φ_b), we have

$$\begin{aligned}\varphi\left[s\epsilon\right] &= \varphi\left[\frac{\epsilon}{s^2}\right]\\ &\leq \lim_{i\to+\infty}\sup\check{d}_b(\widehat{\ell_{\delta_i+1}},\widehat{\ell_{\kappa_i+1}})\\ &\leq \lim_{i\to+\infty}\varphi\left[\xi d_b(\widehat{\ell_{\delta_i}},\widehat{\ell_{\kappa_i}})\right]\\ &= \varphi\left[\xi\left(s\epsilon\right)\right]\\ &< \varphi\left[s\epsilon\right],\end{aligned}$$

which is a contradiction. Therefore, we deduce that $\{\widehat{\ell_i}\}$ is a Cauchy sequence in the closed ball $\overline{b}(\widehat{\ell_0},\rho)$. Since $\overline{b}(\widehat{\ell_0},\rho)$ is closed in $\hat{U}$, there exists a $\sigma^*\in\overline{b}(\widehat{\ell_0},\rho)$ such that $\widehat{\ell_i}\to\sigma^*$. Note that $\sigma^*\in\Lambda$, since $\widehat{\ell_{i+1}}\in\widetilde{T\widehat{\ell_i}}\cap\Lambda$. Next, we claim that

$$\frac{1}{2s}\min\left\{d_b(\widehat{\ell_i},\widetilde{T\widehat{\ell_i}}\cap\Lambda),d_b(\sigma^*,\widetilde{T}\sigma^*\cap\Lambda)\right\}<d_b(\widehat{\ell_i},\sigma^*), \tag{2.17}$$

or

$$\frac{1}{2s}\min\left\{\check{d}_b(\sigma^*,\widetilde{T}\sigma^*\cap\Lambda),d_b(\widehat{\ell}_{i+1},\widetilde{T\widehat{\ell}_{i+1}}\cap\Lambda)\right\}<d_b(\widehat{\ell}_{i+1},\sigma^*)$$

for all $i\in\mathbb{N}$. Assume, on contrary, there exists $i'\in\mathbb{N}$ such that

$$\frac{1}{2s}\min\left\{d_b(\widehat{\ell}_{i'},\widetilde{T\widehat{\ell}_{i'}}\cap\Lambda),\check{d}_b(\sigma^*,\widetilde{T}\sigma^*\cap\Lambda)\right\}\geq d_b(\widehat{\ell}_{i'},\sigma^*) \tag{2.18}$$

and

$$\frac{1}{2s}\min\left\{d_b(\sigma^*,\widetilde{T}\sigma^*\cap\Lambda),d_b(\widehat{\ell}_{i'+1},\widetilde{T\widehat{\ell}_{i'+1}}\cap\Lambda)\right\}\geq d_b(\widehat{\ell}_{i'+1},\sigma^*). \tag{2.19}$$

By (2.18), we have

$$\begin{aligned}
2s\check{d}_b(\widehat{\ell}_{i'},\sigma^*) &\leq \min\left\{d_b(\widehat{\ell}_{i'},\widetilde{T\widehat{\ell}_{i'}}\cap\Lambda),d_b(\sigma^*,\widetilde{T}\sigma^*\cap\Lambda)\right\}\\
&\leq \min\left\{s\left[d_b(\widehat{\ell}_{i'},\sigma^*)+d_b(\sigma^*,\widetilde{T\widehat{\ell}_{i'}}\cap\Lambda)\right],\check{d}_b(\sigma^*,\widetilde{T}\sigma^*\cap\Lambda)\right\}\\
&\leq s\left[d_b(\widehat{\ell}_{i'},\sigma^*)+d_b(\sigma^*,\widetilde{T\widehat{\ell}_{i'}}\cap\Lambda)\right]\\
&< s\left[d_b(\widehat{\ell}_{i'},\sigma^*)+\check{d}_b(\sigma^*,\widetilde{T\widehat{\ell}_{i'}})\right]\\
&\leq s\left[d_b(\widehat{\ell}_{i'},\sigma^*)+d_b(\sigma^*,\widehat{\ell}_{i'+1})\right],
\end{aligned}$$

which implies that

$$d_b(\widehat{\ell}_{i'},\sigma^*)\leq d_b(\sigma^*,\widehat{\ell}_{i'+1}).$$

This together with (2.19) implies

$$\begin{aligned}
d_b(\widehat{\ell}_{i'},\sigma^*) &\leq d_b(\sigma^*,\widehat{\ell}_{i'+1}) \\
&\leq \frac{1}{2s}\min\left\{d_b(\sigma^*,\widetilde{T}\sigma^*\cap\Lambda),d_b(\widehat{\ell}_{i'+1},\widetilde{T\widehat{\ell}_{i'+1}}\cap\Lambda)\right\}.
\end{aligned} \tag{2.20}$$

So

$$\frac{1}{2s}\min\left\{d_b(\widehat{\ell}_{i'},\widetilde{T\widehat{\ell}_{i'}}\cap\Lambda),\check{d}_b(\widehat{\ell}_{i'+1},\widetilde{T\widehat{\ell}_{i'+1}}\cap\Lambda)\right\}<d_b(\widehat{\ell}_{i'},\widehat{\ell}_{i'+1}).$$

From the contractive condition (2.1), we have

$$0<\varphi\left[d_b(\widehat{\ell}_{i'+1},\widehat{\ell}_{i'+2})\right]\leq\varphi\left[H_b(\widetilde{T\widehat{\ell}_{i'}}\cap\Lambda,\widetilde{T\widehat{\ell}_{i'+1}}\cap\Lambda)\right]\leq\varphi\left[\xi(c(\widehat{\ell}_{i'},\widehat{\ell}_{i'+1})))\right],$$

where

$$\begin{aligned}
\Omega(\widehat{\ell}_{i'},\widehat{\ell}_{i'+1}) &= \max\left\{\begin{array}{c} d_b(\widehat{\ell}_{i'},\widehat{\ell}_{i'+1}),d_b(\widehat{\ell}_{i'},\widetilde{T\widehat{\ell}_{i'}}),\check{d}_b(\widehat{\ell}_{i'+1},\widetilde{T\widehat{\ell}_{i'+1}}),\\ \frac{d_b(\widehat{\ell}_{i'},\widetilde{T\widehat{\ell}_{i'+1}})+d_b(\widehat{\ell}_{i'+1},\widetilde{T\widehat{\ell}_{i'}})}{2s}\end{array}\right\}\\
&\leq \max\left\{\begin{array}{c} d_b(\widehat{\ell}_{i'},\widehat{\ell}_{i'+1}),\check{d}_b(\widehat{\ell}_{i'+1},\widehat{\ell}_{i'+2}),\\ \frac{d_b(\widehat{\ell}_{i'},\widehat{\ell}_{i'+2})}{2s}\end{array}\right\}\\
&\leq \max\left\{d_b(\widehat{\ell}_{i'},\widehat{\ell}_{i'+1}),d_b(\widehat{\ell}_{i'+1},\widehat{\ell}_{i'+2})\right\},
\end{aligned}$$

which yields

$$\varphi\left[\check{d}_b(\widehat{\ell}_{i'+1},\widehat{\ell}_{i'+2})\right]\leq\varphi\left[H_b(\widetilde{T\widehat{\ell}_{i'}}\cap\Lambda,\widetilde{T\widehat{\ell}_{i'+1}}\cap\Lambda)\right]<\varphi\left[d_b(\widehat{\ell}_{i'},\widehat{\ell}_{i'+1}))\right].$$

Let $\Delta = \max\left\{d_b(\widehat{\ell_{i'}}, \widehat{\ell_{i'+1}}), d_b(\widehat{\ell_{i'+1}}, \widehat{\ell_{i'+2}})\right\}$. Assume that $\Delta = d_b(\widehat{\ell_{i'+1}}, \widehat{\ell_{i'+2}})$. Since $\widehat{\ell_{i'+2}} \in \widetilde{T\widehat{\ell_{i'+1}}} \cap \Lambda$, we have

$$\varphi\left[d_b(\widehat{\ell_{i'+1}}, \widehat{\ell_{i'+2}})\right] \leq \varphi\left[H_b(\widetilde{T\widehat{\ell_{i'}}} \cap \Lambda, \widetilde{T\widehat{\ell_{i'+1}}} \cap \Lambda)\right] < \varphi\left[d_b(\widehat{\ell_{i'+1}}, \widehat{\ell_{i'+2}}))\right],$$

which is a contradiction. Owing to (φ_a), we have

$$\check{d}_b(\widehat{\ell_{i'+1}}, \widehat{\ell_{i'+2}}) < d_b(\widehat{\ell_{i'}}, \widehat{\ell_{i'+1}}). \tag{2.21}$$

From (2.19), (2.20) and (2.21), we obtain

$$\begin{array}{rcl} d_b(\widehat{\ell_{i'+1}}, \widehat{\ell_{i'+2}}) & < & d_b(\widehat{\ell_{i'}}, \widehat{\ell_{i'+1}}) \\ & \leq & s\left[\check{d}_b(\widehat{\ell_{i'}}, \sigma^*) + d_b(\sigma^*, \widehat{\ell_{i'+1}})\right] \\ & \leq & \left[\begin{array}{l} \frac{1}{2}\min\left\{d_b(\sigma^*, \widetilde{T}\sigma^* \cap \Lambda), d_b(\widehat{\ell_{i'+1}}, \widetilde{T\widehat{\ell_{i'+1}}} \cap \Lambda)\right\} \\ +\frac{1}{2}\min\left\{\check{d}_b(\sigma^*, \widetilde{T}\sigma^* \cap \Lambda), d_b(\widehat{\ell_{i'+1}}, \widetilde{T\widehat{\ell_{i'+1}}} \cap \Lambda)\right\} \end{array}\right] \\ & \leq & \min\left\{d_b(\sigma^*, \widetilde{T}\sigma^* \cap \Lambda), d_b(\widehat{\ell_{i'+1}}, \widehat{\ell_{i'+2}})\right\} \\ & = & d_b(\widehat{\ell_{i'+1}}, \widehat{\ell_{i'+2}}), \end{array}$$

which is a contradiction. Hence (2.17) holds true, that is,

$$\frac{1}{2s}\min\left\{\check{d}_b(\widehat{\ell_i}, \widetilde{T\widehat{\ell_i}} \cap \Lambda), d_b(\sigma^*, \widetilde{T}\sigma^* \cap \Lambda)\right\} < d_b(\widehat{\ell_i}, \sigma^*) \text{ for all } i \geq 2. \tag{2.22}$$

Owing to (2.22), we have

$$\frac{1}{2s}\min\left\{d_b(\widehat{\ell_i}, \widetilde{T\widehat{\ell_i}} \cap \Lambda), d_b(\widehat{\ell_{i+1}}, \widetilde{T\widehat{\ell_{i+1}}} \cap \Lambda)\right\} < d_b(\widehat{\ell_i}, \widehat{\ell_{i+1}}).$$

Moreover, we know that $d_b(\widehat{\ell_i}, \widehat{\ell_{i+1}}) \in E$ for all i. Thus, from (2.1), we have

$$\begin{array}{rcl} \varphi\left[d_b(\widehat{\ell_{i+1}}, \widetilde{T\widehat{\ell_{i+1}}} \cap \Lambda)\right] & \leq & \varphi[H_b(\widetilde{T\widehat{\ell_i}} \cap \Lambda, \widetilde{T\widehat{\ell_{i+1}}} \cap \Lambda)] \\ & \leq & \varphi\left[\xi(\Omega(\widehat{\ell_i}, \widehat{\ell_{i+1}})))\right] \\ & < & \varphi\left[\Omega(\widehat{\ell_i}, \widehat{\ell_{i+1}})))\right], \end{array}$$

where

$$\begin{array}{rcl} \Omega(\widehat{\ell_i}, \widehat{\ell_{i+1}}) & = & \max\left\{\begin{array}{c} d_b(\widehat{\ell_i}, \widehat{\ell_{i+1}}), d_b(\widehat{\ell_i}, \widetilde{T\widehat{\ell_i}}), d_b(\widehat{\ell_{i+1}}, \widetilde{T\widehat{\ell_{i+1}}}), \\ \frac{d_b(\widehat{\ell_i}, \widetilde{T\widehat{\ell_{i+1}}}) + d_b(\widehat{\ell_{i+1}}, \widetilde{T\widehat{\ell_i}})}{2s} \end{array}\right\} \\ & \leq & \max\left\{\begin{array}{c} d_b(\widehat{\ell_i}, \widehat{\ell_{i+1}}), d_b(\widehat{\ell_{i+1}}, \widehat{\ell_{i+2}}), \\ \frac{d_b(\widehat{\ell_i}, \widehat{\ell_{i+2}})}{2s} \end{array}\right\} \\ & \leq & \max\left\{d_b(\widehat{\ell_i}, \widehat{\ell_{i+1}}), d_b(\widehat{\ell_{i+1}}, \widehat{\ell_{i+2}})\right\}, \end{array}$$

which implies

$$\varphi\left[d_b(\widehat{\ell_{i+1}}, \widehat{\ell_{i+2}})\right] \leq \varphi\left[H_b(\widetilde{T\widehat{\ell_i}} \cap \Lambda, \widetilde{T\widehat{\ell_{i+1}}} \cap \Lambda)\right] < \varphi\left[d_b(\widehat{\ell_i}, \widehat{\ell_{i+1}}))\right].$$

Let $\Delta = \max\left\{d_b(\widehat{\ell_i}, \widehat{\ell_{i+1}}), d_b(\widehat{\ell_{i+1}}, \widehat{\ell_{i+2}})\right\}$. Assume that $\Delta = d_b(\widehat{\ell_{i+1}}, \widehat{\ell_{i+2}})$. Since $\widehat{\ell_{i+2}} \in \widetilde{T}\widehat{\ell_{i+1}} \cap \Lambda$, we have

$$\varphi\left[d_b(\widehat{\ell_{i+1}}, \widehat{\ell_{i+2}})\right] \leq \varphi\left[H_b(\widetilde{T}\widehat{\ell_i} \cap \Lambda, \widetilde{T}\widehat{\ell_{i+1}} \cap \Lambda)\right] < \varphi\left[d_b(\widehat{\ell_{i+1}}, \widehat{\ell_{i+2}}))\right],$$

which is a contradiction. Also, by (φ_a), we deduce that

$$d_b(\widehat{\ell_{i+1}}, \widetilde{T}\widehat{\ell_{i+1}} \cap \Lambda) < d_b(\widehat{\ell_i}, \widehat{\ell_{i+1}}). \tag{2.23}$$

Taking the limit $i \to +\infty$ in (2.23), we get

$$\lim_{i\to+\infty} d_b(\widehat{\ell_{i+1}}, \widetilde{T}\widehat{\ell_{i+1}} \cap \Lambda) = 0.$$

Since $g(\widehat{\ell}) = d_b(\widehat{\ell}, \widetilde{T}\widehat{\ell} \cap \Lambda)$ is $\widetilde{T}$-o.l.s.c at σ^*,

$$d_b(\sigma^*, \widetilde{T}\sigma^* \cap \Lambda) = g(\sigma^*) \leq \liminf_i g(\widehat{\ell_{i+1}}) = \liminf_i d_b(\widehat{\ell_{i+1}}, \widetilde{T}\widehat{\ell_{i+1}} \cap \Lambda) = 0.$$

Since $\widetilde{T}\sigma^*$ is closed, we have $\sigma^* \in \widetilde{T}\sigma^*$. Conversely, if σ^* is a fixed point of $\widetilde{T}$ then $g(\sigma^*) = 0 \leq \liminf_i g(\widehat{\ell_i})$, since $\sigma^* \in \Lambda$. □

Corollary 2.3. *Let $(\hat{U}, d_b)$ be a b-metric space with $s \geq 1$, Λ be a closed subset of $\hat{U}$ and ξ be a b-B-GGF on an interval E. A mapping $\widetilde{T} : \Lambda \to CB(\hat{U})$ is said to be a multivalued SU-type φ-contraction if there exists $\varphi \in \varphi^*$ such that for $\widetilde{T}\widehat{\ell} \cap \Lambda \neq \emptyset$*

$$\frac{1}{2s}\min\left\{d_b(\widehat{\ell}, \widetilde{T}\widehat{\ell} \cap \Lambda), d_b(\nu, \widetilde{T}\nu \cap \Lambda)\right\} < d_b(\widehat{\ell}, \nu)$$

implies that

$$\varphi[H_b(\widetilde{T}\widehat{\ell} \cap \Lambda, \widetilde{T}\nu \cap \Lambda)] \leq \varphi[\xi((\widehat{\ell}, \nu)))],$$

for all $\widehat{\ell} \in \Lambda$, $\nu \in \widetilde{T}\widehat{\ell} \cap \Lambda$ with $d_b(\widehat{\ell}, \nu) \in E$, where $H_b(\widetilde{T}\widehat{\ell} \cap \Lambda, \widetilde{T}\nu \cap \Lambda) > 0$. Assume $\widehat{\ell_0} \in \Lambda$ such that $d_b(\widehat{\ell_0}, c^*) \in E$ for some $c^* \in \widetilde{T}\widehat{\ell_0} \cap \Lambda$. Then there exist an orbit $\{\widehat{\ell_i}\}$ of $\widetilde{T}$ in Λ and $\sigma^* \in \Lambda$ such that $\lim_{i\to+\infty} \widehat{\ell_i} = \sigma^*$. Moreover, σ^* is a fixed point of $\widetilde{T}$ if and only if the function $g(\widehat{\ell}) := d_b(\widehat{\ell}, \widetilde{T}\widehat{\ell} \cap \Lambda)$ is $\widetilde{T}$-o.l.s.c at σ^*.

Corollary 2.4. *Let $(\hat{U}, d_b)$ be a b-metric space with $s \geq 1$, Λ be a closed subset of $\hat{U}$ and ξ be a b-B-GGF on an interval E. A mapping $\widetilde{T} : \Lambda \to CB(\hat{U})$ is said to be a multivalued SU-type φ-contraction if there exists $\varphi \in \varphi^*$ such that for $\widetilde{T}\widehat{\ell} \cap \Lambda \neq \emptyset$*

$$\frac{1}{2s}\min\left\{d_b(\widehat{\ell}, \widetilde{T}\widehat{\ell} \cap \Lambda), d_b(\nu, \widetilde{T}\nu \cap \Lambda)\right\} < d_b(\widehat{\ell}, \nu)$$

implies that

$$\varphi[H_b(\widetilde{T}\widehat{\ell} \cap \Lambda, \widetilde{T}\nu \cap \Lambda)] \leq \varphi[\xi(d_b(\widehat{\ell}, \nu)))],$$

for all $\widehat{\ell} \in \hat{U}$, $\nu \in \widetilde{T}\widehat{\ell}$ with $d_b(\widehat{\ell}, \nu) \in E$. Suppose that $\widehat{\ell_0} \in \hat{U}$ such that $d_b(\widehat{\ell_0}, c^*) \in E$ for some $c^* \in \widetilde{T}\widehat{\ell_0}$. Then there exists an orbit $\{\widehat{\ell_i}\}$ of $\widetilde{T}$ in $\hat{U}$ which converges to the fixed point $\sigma^* \in \mathcal{F} = \{\widehat{\ell} \in \hat{U} : d_b(\widehat{\ell}, \sigma^*) \in E\}$ of $\widetilde{T}$.

Example 2.5. Let $\hat{U} = [0, 1]$ be endowed with the metric d_b with coefficient $s \geq \frac{\alpha^2+7}{\alpha^2-1} > 1$ [where $\alpha \geq 3$ is any positive integers] as defined by $d_b(\widehat{\ell}, \nu) = \left|\widehat{\ell} - \nu\right|^2$ for all $\widehat{\ell}, \nu \in \hat{U}$ but not a metric b_d. For $\widehat{\ell}_1 = 0$, $\widehat{\ell}_2 = \frac{1}{2}$ and $\widehat{\ell}_3 = 1$, we obtain

$$b_d\left(\widehat{\ell}_1, \widehat{\ell}_3\right) = 1 > \frac{1}{4} + \frac{1}{4} = b_d\left(\widehat{\ell}_1, \widehat{\ell}_2\right) + b_d\left(\widehat{\ell}_2, \widehat{\ell}_3\right)$$

and let $E = [0, +\infty)$. Consider the mapping $\widetilde{T} : \hat{U} \to CB(\hat{U})$ defined by $\widetilde{T}(\widehat{\ell}) = [0, \widehat{\ell}^2]$. Clearly,

$$\frac{1}{2s} \min\left\{d_b(\widehat{\ell}, \widetilde{T\widehat{\ell}} \cap \Lambda), d_b(\nu, \widetilde{T}\nu \cap \Lambda)\right\} < d_b(\widehat{\ell}, \nu)$$

if and only if $\widehat{\ell}, \nu \in [0, 1]$. Let $\widehat{\ell}_0 = 1$. Then we have $c^* = \frac{1}{2} \in \widetilde{T\widehat{\ell}_0}$ such that $d_b(\widehat{\ell}_0, c^*) \in E$ and

$$\varphi\left[H_b(\widetilde{T\widehat{\ell}}, \widetilde{T}\nu)\right] = \varphi\left[\left|\widehat{\ell}^2 - \nu^2\right|^2\right] \leq \varphi\left[\left|\widehat{\ell} + \nu\right|^2 d_b(\widehat{\ell}, \nu)\right].$$

Set $\varphi(r) = re^r$ for all $r > 0$ and suppose that $\xi(r) = r^2$ is a b-B-GGF of order 2 on $E = [0, \frac{1}{\alpha-1}]$ with coefficient $\frac{\alpha^2+7}{\alpha^2-1}$. For any $\widehat{\ell} \in [0, 1]$ and $\nu \in \widetilde{T\widehat{\ell}}$, we get

$$\varphi\left[H_b(\widetilde{T\widehat{\ell}}, \widetilde{T}\nu)\right] \leq \left[\left|\widehat{\ell} + \nu\right|^2 d_b(\widehat{\ell}, \nu)\right] e^{\left[\left|\widehat{\ell}+\nu\right|^2 d_b(\widehat{\ell}, \nu)\right]} = \varphi\left[\xi(d_b(\widehat{\ell}, \nu))\right].$$

Thus, all the conditions of Corollary 2.3 are fulfilled and 0 is a fixed point of $\widetilde{T}$.

3. Main results

In this section, motivated by the notion of multivalued Suzuki type ∇-contraction, we define the notion of multivalued Suzuki type $(\nabla_\alpha - \xi)$-contraction as follows:

Definition 3.1. Let $(\hat{U}, d_b)$ be a b-metric space with $s \geq 1$, Λ be a closed subset of $\hat{U}$ and ξ be a b-B-GGF on an interval E. A mapping $\widetilde{T} : \Lambda \to CB(\hat{U})$ is said to be a multivalued Suzuki type $(\nabla_\alpha - \xi)$-contraction if there exists $\Gamma \in \nabla$ such that for $\widetilde{T\widehat{\ell}} \cap \Lambda \neq \emptyset$

$$\frac{1}{2s} \min\left\{d_b(\widehat{\ell}, \widetilde{T\widehat{\ell}} \cap \Lambda), d_b(\nu, \widetilde{T}\nu \cap \Lambda)\right\} < d_b(\widehat{\ell}, \nu)$$

implies that

$$\Gamma\left[\alpha(\widehat{\ell}, \nu) H_b(\widetilde{T\widehat{\ell}} \cap \Lambda, \widetilde{T}\nu \cap \Lambda), \xi(\Omega(\widehat{\ell}, \nu))\right] \geq 0, \tag{3.1}$$

where

$$\Omega(\widehat{\ell}, \nu) = \max\left\{d_b(\widehat{\ell}, \nu), d_b(\widehat{\ell}, \widetilde{T\widehat{\ell}}), d_b(\nu, \widetilde{T}\nu), \frac{d_b(\widehat{\ell}, \widetilde{T}\nu) + d_b(\nu, \widetilde{T\widehat{\ell}})}{2s}\right\}$$

for all $\widehat{\ell} \in \Lambda$, $\nu \in \widetilde{T\widehat{\ell}} \cap \Lambda$ with $d_b(\widehat{\ell}, \nu) \in E$.

The second one of our results is as follows.

Theorem 3.2. *Let $(\hat{U}, d_b)$ be a complete b-metric space with $s \geq 1$, Λ be a closed subset of $\hat{U}$ and $\widetilde{T} : \Lambda \to CB(\hat{U})$ be a multivalued SU-type $(\alpha$-$\nabla)$-contraction. Suppose that the following conditions are satisfied:*

(*i*) *$\widetilde{T}$ is α-admissible;*
(*ii*) *there exists $\widehat{\ell_0} \in \Lambda$ with $d_b(\widehat{\ell_0}, \widehat{\ell_1}) \in E$ for some $\widehat{\ell_1} \in \widetilde{T\widehat{\ell_0}} \cap \Lambda$ such that $\alpha\left(\widehat{\ell_0}, \widehat{\ell_1}\right) \geq 1$.*

Then there exist an orbit $\{\widehat{\ell_i}\}$ of $\widetilde{T}$ in Λ and $\sigma^* \in \Lambda$ such that $\lim_{i\to+\infty} \widehat{\ell_i} = \sigma^*$. Moreover, σ^* is a fixed point of $\widetilde{T}$ if and only if the function $g(\widehat{\ell}) := d_b(\widehat{\ell}, \widetilde{T\ell} \cap \Lambda)$ is $\widetilde{T}$-o.l.s.c at σ^*.

Proof. Owing to the hypothesis, there exists $\widehat{\ell_0} \in \Lambda$ with $d_b(\widehat{\ell_0}, \widehat{\ell_1}) \in E$ for some $\widehat{\ell_1} \in \widetilde{T\widehat{\ell_0}} \cap \Lambda$ such that $\alpha\left(\widehat{\ell_0}, \widehat{\ell_1}\right) \geq 1$. On the other hand, we have

$$\frac{1}{2s} \min\left\{d_b(\widehat{\ell_0}, \widetilde{T\widehat{\ell_0}} \cap \Lambda), d_b(\widehat{\ell_1}, \widetilde{T\widehat{\ell_1}} \cap \Lambda)\right\} < d_b(\widehat{\ell_0}, \widehat{\ell_1}). \tag{3.2}$$

If $d_b(\widehat{\ell_0}, \widehat{\ell_1}) = 0$, then $\widehat{\ell_0}$ is a fixed point of $\widetilde{T}$. Thus, we assume that $d_b(\widehat{\ell_0}, \widehat{\ell_1}) \neq 0$. Define $\rho = \sigma(d_b(\widehat{\ell_0}, \widehat{\ell_1}))$. From (1.16), we have $\sigma(r) \geq r$. Hence $d_b(\widehat{\ell_0}, \widehat{\ell_1}) \leq \rho$ and so $\widehat{\ell_1} \in \overline{b}(\widehat{\ell_0}, \rho)$. Since $\alpha\left(\widehat{\ell_0}, \widehat{\ell_1}\right) \geq 1$ and $d_b(\widehat{\ell_0}, \widehat{\ell_1}) \in E$, from (3.1) and (3.2), it follows that

$$\begin{aligned} 0 &\leq \Gamma[\alpha(\widehat{\ell_0}, \widehat{\ell_1}) H_b(\widetilde{T\widehat{\ell_0}} \cap \Lambda, \widetilde{T\widehat{\ell_1}} \cap \Lambda), \xi(d_b(\widehat{\ell_0}, \widehat{\ell_1}))] \\ &< \xi(\Omega(\widehat{\ell_0}, \widehat{\ell_1})) - \alpha(\widehat{\ell_0}, \widehat{\ell_1}) H_b(\widetilde{T\widehat{\ell_0}} \cap \Lambda, \widetilde{T\widehat{\ell_1}} \cap \Lambda), \end{aligned}$$

which implies

$$\alpha(\widehat{\ell_0}, \widehat{\ell_1}) H_b(\widetilde{T\widehat{\ell_0}} \cap \Lambda, \widetilde{T\widehat{\ell_1}} \cap \Lambda) < \xi(\Omega(\widehat{\ell_0}, \widehat{\ell_1})).$$

We can choose an $\epsilon_1 > 0$ such that

$$\alpha(\widehat{\ell_0}, \widehat{\ell_1}) H_b(\widetilde{T\widehat{\ell_0}} \cap \Lambda, \widetilde{T\widehat{\ell_1}} \cap \Lambda) + \epsilon_1 \leq \xi(\Omega(\widehat{\ell_0}, \widehat{\ell_1})).$$

Thus

$$\begin{aligned} d_b(\widehat{\ell_1}, \widetilde{T\widehat{\ell_1}} \cap \Lambda) + \epsilon_1 &\leq H_b(\widetilde{T\widehat{\ell_0}} \cap \Lambda, \widetilde{T\widehat{\ell_1}} \cap \Lambda) + \epsilon_1 \\ &\leq \alpha(\widehat{\ell_0}, \widehat{\ell_1}) H_b(\widetilde{T\widehat{\ell_0}} \cap \Lambda, \widetilde{T\widehat{\ell_1}} \cap \Lambda) + \epsilon_1 \\ &\leq \xi(\Omega(\widehat{\ell_0}, \widehat{\ell_1})). \end{aligned} \tag{3.3}$$

It follows from Lemma 1.2 that there exists $\widehat{\ell_2} \in \widetilde{T\widehat{\ell_1}} \cap \Lambda$ such that

$$d_b(\widehat{\ell_1}, \widehat{\ell_2}) \leq d_b(\widehat{\ell_1}, \widetilde{T\widehat{\ell_1}} \cap \Lambda) + \epsilon_1. \tag{3.4}$$

From (3.3) and (3.4), we have

$$d_b(\widehat{\ell_1}, \widehat{\ell_2}) \leq \xi(\Omega(\widehat{\ell_0}, \widehat{\ell_1})),$$

where

$$\Omega(\widehat{\ell_0}, \widehat{\ell_1}) = \max\left\{d_b(\widehat{\ell_0}, \widehat{\ell_1}), d_b(\widehat{\ell_0}, \widetilde{T\widehat{\ell_0}}), d_b(\widehat{\ell_1}, \widetilde{T\widehat{\ell_1}}), \frac{d_b(\widehat{\ell_0}, \widetilde{T\widehat{\ell_1}}) + d_b(\widehat{\ell_1}, \widetilde{T\widehat{\ell_0}})}{2s}\right\}$$

$$\leq \max\left\{d_b(\widehat{\ell_0},\widehat{\ell_1}), d_b(\widehat{\ell_1},\widetilde{T}\widehat{\ell_1}), \frac{d_b(\widehat{\ell_0},\widetilde{T}\widehat{\ell_1})}{2s}\right\}$$
$$\leq \max\left\{d_b(\widehat{\ell_0},\widehat{\ell_1}), d_b(\widehat{\ell_1},\widetilde{T}\widehat{\ell_1})\right\}.$$

We claim that

$$d_b(\widehat{\ell_1},\widehat{\ell_2}) \leq \xi(d_b(\widehat{\ell_0},\widehat{\ell_1})). \tag{3.5}$$

Let $\Delta = \max\left\{d_b(\widehat{\ell_0},\widehat{\ell_1}), d_b(\widehat{\ell_1},\widetilde{T}\widehat{\ell_1})\right\}$. Assume that $\Delta = d_b(\widehat{\ell_1},\widetilde{T}\widehat{\ell_1})$. Since $\widehat{\ell_2} \in \widetilde{T}\widehat{\ell_1} \cap \Lambda$, we have

$$(d_b(\widehat{\ell_1},\widehat{\ell_2}) \leq \xi(d_b(\widehat{\ell_1},\widehat{\ell_2})),$$

which is a contradiction. Hence (3.5) holds true. We assume that $d_b(\widehat{\ell_1},\widehat{\ell_2}) \neq 0$, otherwise, $\widehat{\ell_1}$ is a fixed point of $\widetilde{T}$. Since $d_b(\widehat{\ell_1},\widehat{\ell_2}) \leq \xi(d_b(\widehat{\ell_0},\widehat{\ell_1})) < d_b(\widehat{\ell_0},\widehat{\ell_1})$, we deduce that $d_b(\widehat{\ell_1},\widehat{\ell_2}) \in E$. Next, $\widehat{\ell_2} \in \overline{b}(\widehat{\ell_0},\rho)$ since

$$\begin{aligned}
d_b(\widehat{\ell_0},\widehat{\ell_2}) \leq s d_b(\widehat{\ell_0},\widehat{\ell_1}) + s d_b(\widehat{\ell_1},\widehat{\ell_2}) &\leq s d_b(\widehat{\ell_0},\widehat{\ell_1}) + s^2 d_b(\widehat{\ell_1},\widehat{\ell_2}) \\
&\leq s d_b(\widehat{\ell_0},\widehat{\ell_1}) + s^2 \xi(d_b(\widehat{\ell_0},\widehat{\ell_1})) \\
&= s\left[d_b(\widehat{\ell_0},\widehat{\ell_1}) + s\xi(d_b(\widehat{\ell_0},\widehat{\ell_1}))\right] \\
&\leq s\sigma d_b(\widehat{\ell_0},\widehat{\ell_1}) \\
&\leq d_b(\widehat{\ell_0},\widehat{\ell_1}) + s\sigma(d_b(\widehat{\ell_0},\widehat{\ell_1})) \\
&= \sigma(d_b(\widehat{\ell_0},\widehat{\ell_1})) = \rho..
\end{aligned}$$

Since $\widetilde{T}$ is α-admissible, $\alpha\left(\widehat{\ell_1},\widehat{\ell_2}\right) \geq 1$. Also, since

$$\frac{1}{2s}\min\left\{d_b(\widehat{\ell_1},\widetilde{T}\widehat{\ell_1} \cap \Lambda), d_b(\widehat{\ell_2},\widetilde{T}\widehat{\ell_2} \cap \Lambda)\right\} < d_b(\widehat{\ell_1},\widehat{\ell_2}),$$

from the contractive condition (3.1), we get

$$\begin{aligned}
&0 \leq \Gamma[\alpha(\widehat{\ell_1},\widehat{\ell_2})H_b(\widetilde{T}\widehat{\ell_1} \cap \Lambda, \widetilde{T}\widehat{\ell_2} \cap \Lambda), \xi(\Omega(\widehat{\ell_1},\widehat{\ell_2}))] \\
&\quad < \xi(\Omega(\widehat{\ell_1},\widehat{\ell_2})) - \alpha(\widehat{\ell_1},\widehat{\ell_2})H_b(\widetilde{T}\widehat{\ell_1} \cap \Lambda, \widetilde{T}\widehat{\ell_2} \cap \Lambda).
\end{aligned}$$

This implies that

$$\alpha(\widehat{\ell_1},\widehat{\ell_2})H_b(\widetilde{T}\widehat{\ell_1} \cap \Lambda, \widetilde{T}\widehat{\ell_2} \cap \Lambda) < \xi(\Omega(\widehat{\ell_1},\widehat{\ell_2})).$$

Now choose an $\epsilon_2 > 0$ such that

$$\alpha(\widehat{\ell_1},\widehat{\ell_2})H_b(\widetilde{T}\widehat{\ell_1} \cap \Lambda, \widetilde{T}\widehat{\ell_2} \cap \Lambda) + \epsilon_2 \leq \xi(\Omega(\widehat{\ell_1},\widehat{\ell_2})).$$

Thus,

$$\begin{aligned}
d_b(\widehat{\ell_2},\widetilde{T}\widehat{\ell_2} \cap \Lambda) + \epsilon_2 &\leq H_b(\widetilde{T}\widehat{\ell_1} \cap \Lambda, \widetilde{T}\widehat{\ell_2} \cap \Lambda) + \epsilon_2 \\
&\leq \alpha(\widehat{\ell_1},\widehat{\ell_2})H_b(\widetilde{T}\widehat{\ell_1} \cap \Lambda, \widetilde{T}\widehat{\ell_2} \cap \Lambda) + \epsilon_2 \\
&\leq \xi(\Omega(\widehat{\ell_1},\widehat{\ell_2})).
\end{aligned} \tag{3.6}$$

It follows from Lemma 1.2 that there exists $\widehat{\ell_3} \in \widetilde{T\widehat{\ell_2}} \cap \Lambda$ such that

$$d_b(\widehat{\ell_2}, \widehat{\ell_3}) \leq d_b(\widehat{\ell_2}, \widetilde{T\widehat{\ell_2}} \cap \Lambda) + \epsilon_2. \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$d_b(\widehat{\ell_2}, \widehat{\ell_3}) \leq \xi(\Omega(\widehat{\ell_1}, \widehat{\ell_2})),$$

where

$$\begin{aligned}
\Omega(\widehat{\ell_1}, \widehat{\ell_2}) &= \max\left\{d_b(\widehat{\ell_1}, \widehat{\ell_2}), d_b(\widehat{\ell_1}, \widetilde{T\widehat{\ell_1}}), d_b(\widehat{\ell_2}, \widetilde{T\widehat{\ell_2}}), \frac{d_b(\widehat{\ell_1}, \widetilde{T\widehat{\ell_2}}) + d_b(\widehat{\ell_2}, \widetilde{T\widehat{\ell_1}})}{2s}\right\} \\
&\leq \max\left\{d_b(\widehat{\ell_1}, \widehat{\ell_2}), d_b(\widehat{\ell_2}, \widetilde{T\widehat{\ell_2}}), \frac{d_b(\widehat{\ell_1}, \widetilde{T\widehat{\ell_2}})}{2s}\right\} \\
&\leq \max\left\{d_b(\widehat{\ell_1}, \widehat{\ell_2}), d_b(\widehat{\ell_2}, \widetilde{T\widehat{\ell_2}})\right\}.
\end{aligned}$$

This implies that

$$d_b(\widehat{\ell_2}, \widehat{\ell_3}) \leq \xi d_b(\widehat{\ell_1}, \widehat{\ell_2})). \tag{3.8}$$

Let $\Delta = \max\left\{d_b(\widehat{\ell_1}, \widehat{\ell_2}), d_b(\widehat{\ell_2}, \widetilde{T\widehat{\ell_2}})\right\}$. Assume that $\Delta = d_b(\widehat{\ell_2}, \widetilde{T\widehat{\ell_2}})$. Since $\widehat{\ell_3} \in \widetilde{T\widehat{\ell_2}} \cap \Lambda$, we have

$$d_b(\widehat{\ell_2}, \widehat{\ell_3}) \leq \xi \widehat{d(\widehat{\ell_2}, \widehat{\ell_3})}),$$

which is a contradiction. Hence (3.8) holds true. We assume that $d_b(\widehat{\ell_2}, \widehat{\ell_3}) \neq 0$, otherwise, $\widehat{\ell_2}$ is a fixed point of $\widetilde{T}$. From (3.8), we have $d_b(\widehat{\ell_2}, \widehat{\ell_3}) < d_b(\widehat{\ell_1}, \widehat{\ell_2})$ and so $d_b(\widehat{\ell_2}, \widehat{\ell_3}) \in E$. Also, we have $\widehat{\ell_3} \in \overline{b}(\widehat{\ell_0}, \rho)$, since

$$\begin{aligned}
d_b(\widehat{\ell_0}, \widehat{\ell_3}) \leq s d_b(\widehat{\ell_0}, \widehat{\ell_1}) + s^2 d_b(\widehat{\ell_1}, \widehat{\ell_2}) + s^3 d_b(\widehat{\ell_2}, \widehat{\ell_3}) &= s\left[d_b(\widehat{\ell_0}, \widehat{\ell_1}) + s d_b(\widehat{\ell_1}, \widehat{\ell_2}) + s^2 d_b(\widehat{\ell_2}, \widehat{\ell_3})\right] \\
&\leq s\left[d_b(\widehat{\ell_0}, \widehat{\ell_1}) + \xi(d_b(\widehat{\ell_0}, \widehat{\ell_1})) + \xi^2(d_b(\widehat{\ell_0}, \widehat{\ell_1}))\right] \\
&\leq s\sigma d_b(\widehat{\ell_0}, \widehat{\ell_1}) \\
&\leq d_b(\widehat{\ell_0}, \widehat{\ell_1}) + s\sigma(d_b(\widehat{\ell_0}, \widehat{\ell_1})) \\
&= \sigma(d_b(\widehat{\ell_0}, \widehat{\ell_1})) = \rho.
\end{aligned}$$

Continuing this manner, we obtain a sequence $\left\{\widehat{\ell_i}\right\} \subset \overline{b}(\widehat{\ell_0}, \rho)$ such that $\widehat{\ell_{i+1}} \in \widetilde{T\widehat{\ell_i}} \cap \Lambda$, $\widehat{\ell_i} \neq \widehat{\ell_{i+1}}$ with $\alpha\left(\widehat{\ell_i}, \widehat{\ell_{i+1}}\right) \geq 1$, $d_b(\widehat{\ell_i}, \widehat{\ell_{i+1}}) \in E$ and by the above hypothesis, we have

$$d_b(\widehat{\ell_i}, \widehat{\ell_{i+1}}) \leq \xi^i(d_b(\widehat{\ell_0}, \widehat{\ell_1})), \quad \text{for all } i \in \mathbb{N}. \tag{3.9}$$

For any $q \in \mathbb{N}$, by using the triangular inequality and (3.9), we get

$$\begin{aligned}
d_b(\widehat{\ell_i}, \widehat{\ell_{i+q}}) &\leq s^i d_b(\widehat{\ell_i}, \widehat{\ell_{i+1}}) + s^{i+1} d_b(\widehat{\ell_{i+1}}, \widehat{\ell_{i+2}}) + \cdots + s^{i+q-1} d_b(\widehat{\ell_{i+q-1}}, \widehat{\ell_{i+q}}) \\
&\leq s^i \xi^i(d_b(\widehat{\ell_0}, \widehat{\ell_1})) + s^{i+1}\xi^{i+1}(d_b(\widehat{\ell_0}, \widehat{\ell_1})) + \cdots + s^{i+q-1}\xi^{i+q-1}(d_b(\widehat{\ell_0}, \widehat{\ell_1})) \\
&\leq \sum_{j=i}^{\infty} s^j \xi^j(d_b(\widehat{\ell_0}, \widehat{\ell_1})) < \infty.
\end{aligned} \tag{3.10}$$

Assume that

$$H_i = \sum_{j=i}^{\infty} s^j \xi^j (d_b(\widehat{\ell_0}, \widehat{\ell_1})) \text{ and } \lim_{i\to+\infty} H_i = H. \tag{3.11}$$

By (3.10) and (3.11), we get

$$d_b(\widehat{\ell_i}, \widehat{\ell_{i+q}}) \leq \left(H_{i+q-1} - H_i\right). \tag{3.12}$$

Due to (3.11), (3.12) implies that $d_b(\widehat{\ell_i}, \widehat{\ell_{i+q}}) \to 0$ as $i \to +\infty$. Hence $\{\widehat{\ell_i}\}$ is a Cauchy sequence in the closed ball $\overline{b}(\widehat{\ell_0}, \rho)$. Since $\overline{b}(\widehat{\ell_0}, \rho)$ is closed in $\hat{U}$, there exists an $\sigma^* \in \overline{b}(\widehat{\ell_0}, \rho)$ such that $\widehat{\ell_i} \to \sigma^*$. Note that $\sigma^* \in \Lambda$, since $\widehat{\ell_{i+1}} \in \widetilde{T}\widehat{\ell_i} \cap \Lambda$. By the same argument as in Theorem 2.2, we have

$$\frac{1}{2s} \min\left\{d_b(\widehat{\ell_i}, \widetilde{T}\widehat{\ell_i} \cap \Lambda), d_b(\widehat{\ell_{i+1}}, \widetilde{T}\widehat{\ell_{i+1}} \cap \Lambda)\right\} < d_b(\widehat{\ell_i}, \widehat{\ell_{i+1}}).$$

Also, we know that $\alpha\left(\widehat{\ell_i}, \widehat{\ell_{i+1}}\right) \geq 1$ and $d_b(\widehat{\ell_i}, \widehat{\ell_{i+1}}) \in E$ for all n. Thus, from (3.1), we have

$$\begin{aligned} 0 &\leq \Gamma[\alpha(\widehat{\ell_i}, \widehat{\ell_{i+1}}) H_b(\widetilde{T}\widehat{\ell_i} \cap \Lambda, \widetilde{T}\widehat{\ell_{i+1}} \cap \Lambda), \xi(\Omega(\widehat{\ell_i}, \widehat{\ell_{i+1}}))] \\ &< \xi(\Omega(\widehat{\ell_i}, \widehat{\ell_{i+1}})) - \alpha(\widehat{\ell_i}, \widehat{\ell_{i+1}}) H_b(\widetilde{T}\widehat{\ell_i} \cap \Lambda, \widetilde{T}\widehat{\ell_{i+1}} \cap \Lambda), \end{aligned}$$

which gives that

$$\alpha(\widehat{\ell_i}, \widehat{\ell_{i+1}}) H_b(\widetilde{T}\widehat{\ell_i} \cap \Lambda, \widetilde{T}\widehat{\ell_{i+1}} \cap \Lambda) < \xi(\Omega(\widehat{\ell_i}, \widehat{\ell_{i+1}})).$$

Since $\widehat{\ell_{i+1}} \in \widetilde{T}\widehat{\ell_i} \cap \Lambda$, from (3.9), we get

$$\begin{aligned} d_b(\widehat{\ell_{i+1}}, \widetilde{T}\widehat{\ell_{i+1}} \cap \Lambda) &\leq \alpha(\widehat{\ell_i}, \widehat{\ell_{i+1}}) H_b(\widetilde{T}\widehat{\ell_i} \cap \Lambda, \widetilde{T}\widehat{\ell_{i+1}} \cap \Lambda) \\ &< \xi(d_b(\widehat{\ell_i}, \widehat{\ell_{i+1}})) \\ &\leq \xi^{i+1}(d_b(\widehat{\ell_0}, \widehat{\ell_1})). \end{aligned} \tag{3.13}$$

Taking the limit $i \to +\infty$ in (3.13), we obtain

$$\lim_{i\to+\infty} d_b(\widehat{\ell_{i+1}}, \widetilde{T}\widehat{\ell_{i+1}} \cap \Lambda) = 0.$$

Since $g(\widehat{\ell}) = d_b(\widehat{\ell}, \widetilde{T}\widehat{\ell} \cap \Lambda)$ is $\widetilde{T}$-orbitally lower semi-continuous at σ^*,

$$d_b(\sigma^*, \widetilde{T}\sigma^* \cap \Lambda) = g(\sigma^*) \leq \liminf_i g(\widehat{\ell_{i+1}}) = \liminf_i \check{d}_b(\widehat{\ell_{i+1}}, \widetilde{T}\widehat{\ell_{i+1}} \cap \Lambda) = 0.$$

Since $\widetilde{T}\sigma^*$ is closed, we have $\sigma^* \in \widetilde{T}\sigma^*$. Conversely, if σ^* is a fixed point of $\widetilde{T}$ then $g(\sigma^*) = 0 \leq \liminf_i g(\widehat{\ell_i})$, since $\sigma^* \in \Lambda$. □

Setting $\Gamma(r, s) = s - \int_0^r \varsigma(t)dt$ for all $r, s \geq 0$ in Theorem 3.2, we get the following result.

Corollary 3.3. *Let $(\hat{U}, d_b)$ be a complete b-metric space with $s \geq 1$, Λ be a closed subset of $\hat{U}$, ξ be a b-B-GGF on an interval E and let $\widetilde{T} : \Lambda \to CB(\hat{U})$ be a given multivalued mapping. Suppose that for $\widetilde{T}\widehat{\ell} \cap \Lambda \neq \emptyset$ such that*

$$\frac{1}{2s} \min\left\{d_b(\widehat{\ell}, \widetilde{T}\widehat{\ell} \cap \Lambda), d_b(\nu, \widetilde{T}\nu \cap \Lambda)\right\} < d_b(\widehat{\ell}, \nu)$$

implies that

$$\int_0^{\alpha(\widehat{\ell},\nu)H_b(\widetilde{T}\widehat{\ell}\cap\Lambda,\widetilde{T}\nu\cap\Lambda)}\varsigma(t)dt \leq \xi(\widehat{d(\widehat{\ell},\nu)})$$

for all $\widehat{\ell} \in \Lambda$, $\nu \in \widetilde{T\widehat{\ell}} \cap \Lambda$ *with* $\widehat{d(\widehat{\ell},\nu)} \in E$*, where* $\varsigma : \mathbb{R}^+ \to \mathbb{R}^+$ *is a function such that* $\int_0^{\epsilon}\varsigma(t)dt$ *exists and* $\int_0^{\epsilon}\varsigma(t)dt > \epsilon$ *for all* $\epsilon > 0$*. Suppose that the following conditions are satisfied:*

(*i*) $\widetilde{T}$ *is* α*-admissible;*
(*ii*) *there exists* $\widehat{\ell_0} \in \Lambda$ *with* $d_b(\widehat{\ell_0},\widehat{\ell_1}) \in E$ *for some* $\widehat{\ell_1} \in \widetilde{T\widehat{\ell_0}} \cap \Lambda$ *such that* $\alpha\left(\widehat{\ell_0},\widehat{\ell_1}\right) \geq 1$.

Then there exist an orbit $\{\widehat{\ell_i}\}$ *of* $\widetilde{T}$ *in* Λ *and* $\sigma^* \in \Lambda$ *such that* $\lim_{i\to+\infty}\widehat{\ell_i} = \sigma^*$*. Moreover,* σ^* *is a fixed point of* $\widetilde{T}$ *if and only if the function* $g(\widehat{\ell}) := d_b(\widehat{\ell},\widetilde{T\widehat{\ell}} \cap \Lambda)$ *is* $\widetilde{T}$*-o.l.s.c at* σ^**.*

Corollary 3.4. *Let* $(\hat{U}, d_b)$ *be a complete b-metric space with* $s \geq 1$*,* ξ *be b -B-GGF on an interval E and let* $\widetilde{T} : \hat{U} \to CB(\hat{U})$ *be a given multivalued mapping. Suupose that there exist* $\psi \in \Phi$ *and* $\Gamma \in \nabla$ *such that*

$$\frac{1}{2s}\min\left\{d_b(\widehat{\ell},\widetilde{T\widehat{\ell}} \cap \Lambda), \check{d}_b(\nu,\widetilde{T}\nu \cap \Lambda)\right\} < d_b(\widehat{\ell},\nu)$$

implies that

$$\Gamma\left[\alpha(\widehat{\ell},\nu)H_b(\widetilde{T\widehat{\ell}},\widetilde{T}\nu), \xi(d_b(\widehat{\ell},\nu))\right] \geq 0$$

for all $\widehat{\ell} \in \hat{U}$, $\nu \in \widetilde{T\widehat{\ell}}$ *with* $d_b(\widehat{\ell},\nu) \in E$*. Suppose that the following conditions are satisfied:*

(*i*) $\widetilde{T}$ *is* α*-admissible;*
(*ii*) *there exists* $\widehat{\ell_0} \in \hat{U}$ *with* $d_b(\widehat{\ell_0},\widehat{\ell_1}) \in E$ *for some* $\widehat{\ell_1} \in \widetilde{T\widehat{\ell_0}}$ *such that*$\alpha\left(\widehat{\ell_0},\widehat{\ell_1}\right) \geq 1$.

Then there exists an orbit $\{\widehat{\ell_i}\}$ *of* $\widetilde{T}$ *in* $\hat{U}$ *which converges to the fixed point* $\sigma^* \in \mathcal{F} = \{\widehat{\ell} \in \hat{U} : d_b(\widehat{\ell},\sigma^*) \in E\}$ *of* $\widetilde{T}$*.*

4. An application

In the recent past, Banach's fixed point theorem has a broad family of important applications to an iteration methods for the system of linear algebraic equation and the most publicized application of Banach's fixed point theorem emarge in the module of function spaces. This yields the existence of solution for the system of differential and integral equations (see [3]). In this section, we investigate Corollary 2.4 to stabilize the existence of solution for the system of integral inclusions.

Consider the following system of integral inclusion:

$$\varsigma(r) \in \kappa + U\int_{r_0}^{r} D(t,\varsigma(t))\,dt, \tag{4.1}$$

where $\kappa \in (-\infty,+\infty)$, U is a bounded compact subset of $(-\infty,+\infty)$ and the operator $D(t,\varsigma(t))$ is lower semi-continuous. Let $\hat{U} = C(I)$ be the space of all continuous real valued functions ($C(I)$ is complete with respect to the metric d_b) endowed with the b-metric defined by

$$d_b\left(\widehat{\ell_1},\widehat{\ell_2}\right) = \sup_{r\in I}\left|\widehat{\ell_1}(r) - \widehat{\ell_2}(r)\right|.$$

Assume that there exists $D : (-\infty, +\infty) \times (-\infty, +\infty) \to (-\infty, +\infty)$ which is continuous on

$$\Gamma = \left\{ (r, \varsigma) : \ |r - r_0| \leq \left[\frac{\alpha_1^{h-2}}{\alpha_1^{h-1}} \right] \text{ and } |\varsigma - \kappa| \leq \frac{1}{2} \left(\frac{\alpha_2}{\alpha_1} \right) \right\}$$

where $\alpha_1 = \max_{u \in U} |U|$, $0 < \alpha_2 < \alpha_1$ and $h \geq 2$ such that

$$|D(r, \varsigma_1(r)) - D(r, \varsigma_2(r))| \leq \frac{\alpha_1}{\alpha_2} |\varsigma_1(r) - \varsigma_2(r)|^h ,$$

where D is bounded as

$$|D(t, \varsigma)| < \frac{1}{2} \left[\frac{\alpha_2}{\alpha_1} \right]^h .$$

Moreover, let $\check{C} = \left\{ \varsigma \in C(I) : \widehat{V}(\varsigma, \kappa) \leq \frac{1}{2\alpha_2} \right\}$ be a closed subspace of $C(I)$ and the operator g be defined by

$$g(\varsigma(r)) \in \kappa + U \int_{r_0}^{r} V(t, \varsigma(t))\, dt.$$

Set $V_{\hat{U}}(r) = \int_{r_0}^{r} V(t, \varsigma(t))\, dt$. Note that

$$\begin{aligned} H_b[g(\varsigma_1(r)), g(\varsigma_2(r))] &= H_b[\kappa + UV_{\hat{U}}(r), \kappa + UV_y(r)] \\ &\leq H_b[UV_{\hat{U}}(r), UV_y(r)] \\ &= \max \left\{ \max_{\overline{a} \in UV_{\hat{U}}(r)} \check{d}_b\left(\overline{a}, UV_y(r)\right), \max_{\overline{b} \in UV_y(r)} d_b\left(\overline{b}, UV_{\hat{U}}(r)\right) \right\}. \end{aligned} \tag{4.2}$$

Then

$$\begin{aligned} \max_{\overline{a} \in UV_{\hat{U}}(r)} d_b\left(\overline{a}, UV_y(r)\right) &= \max_{\overline{a} \in UV_{\hat{U}}(r)} \min_{\overline{b} \in UV_y(r)} d_b\left(\overline{a}, \overline{b}\right) \\ &= \max_{\overline{u} \in U} \min_{\overline{v} \in U} \check{d}_b\left(\overline{u}V(r, \varsigma_1(r)), \overline{v}V(r, \varsigma_2(r))\right) \\ &= \max_{\overline{u} \in U} \min_{\overline{v} \in U} \sup_{r \in I} \ |\overline{u}V(r, \varsigma_1(r)) - \overline{v}V(r, \varsigma_2(r))| \\ &\leq \max_{\overline{u} \in U} \min_{\overline{v} \in U} \sup_{r \in I} [|\overline{u}V(r, \varsigma_2(r)) - \overline{v}V(r, \varsigma_2(r))| \\ &\quad + |\overline{u}V(r, \varsigma_2(r)) - \overline{u}V(r, \varsigma_1(r))|] \\ &\leq \max_{\overline{u} \in U} \min_{\overline{v} \in U} [|\overline{u}| \sup_{r \in I} |V(r, \varsigma_2(r)) - V(r, \varsigma_1(r))| \\ &\quad + |\overline{u} - \overline{v}| \sup_{r \in I} |V(r, \varsigma_2(r))|] \\ &= \max_{\overline{u} \in U} |\overline{u}| \sup_{r \in I} |V(r, \varsigma_2(r)) - V(r, \varsigma_1(r))| \\ &= \alpha_2 \sup_{r \in I} |V(r, \varsigma_2(r)) - V(r, \varsigma_1(r))| . \end{aligned}$$

This implies that

$$\max_{\overline{a} \in UV_{\hat{U}}(r)} d\left(\overline{a}, UV_y(r)\right) \leq \alpha_2 \sup_{r \in I} |V(r, \varsigma_2(r)) - V(r, \varsigma_1(r))| . \tag{4.3}$$

The third one of our results is as follows:

Theorem 4.1. *Let $\hat{U} = C(I)$ be the space of all continuous real valued functions and $g : \left(\check{C}, d\right) \to \left(V\left(\check{C}\right), H_b\right)$ be a lower semi-continuous mapping. Suppose that the following assumptions hold:*
(i) g is defined for all $\varsigma \in \check{C}$;
(ii) $g(\varsigma(r))$ is a compact subset of $\check{C}$ for all $\varsigma \in \check{C}$;
Then the integral equation (4.3) *has a solution on*

$$I = \left[r_0 - \frac{\alpha_1^{h-2}}{\alpha_1^{h-1}}, r_0 + \frac{\alpha_1^{h-2}}{\alpha_1^{h-1}}\right].$$

Proof. Let $\varkappa \in I$. Then $|\varkappa - r_0| \le \left[\frac{\alpha_1^{h-2}}{\alpha_1^{h-1}}\right]$. Hence we have $|\varsigma(\varkappa) - \kappa| \le \frac{1}{2}\left(\frac{\alpha_2}{\alpha_1}\right)$. If $(\varkappa, \varsigma(\varkappa)) \in (-\infty, +\infty)$, then the integral equation in (4.1) exists. Since $\kappa \in (-\infty, +\infty)$ is continuous, $\varkappa$ is defined for all $\varkappa \in \check{C}$. Next, let $\vartheta(r) \in g(\varsigma(r))$. Then $\vartheta(r) = \kappa + \overline{u}V_{\hat{U}}(r)$ for $\overline{u} \in U$ and so

$$\begin{aligned}
|\vartheta(r) - \kappa| &= \left|\overline{u}V_{\hat{U}}(r)\right| = |\overline{u}|\left|V_{\hat{U}}(r)\right| \\
&\le \alpha_1 \int_{r_0}^{r} |V(t, \varsigma(t))\, dt| \\
&\le \alpha_1 \int_{r_0}^{r} |V(t, \varsigma(t))|\, dt \\
&< \alpha_1 \frac{1}{2}\left(\frac{\alpha_2}{\alpha_1}\right)^h \\
&\le \frac{1}{2}\left(\frac{\alpha_2}{\alpha_1}\right).
\end{aligned}$$

Thus $|\vartheta(r) - \kappa| \le \frac{1}{2}\left(\frac{\alpha_2}{\alpha_1}\right)$ for all $\vartheta(r) \in g(\varsigma(r))$. So $g(\varsigma(r))$ is a subset of $\check{C}$. Now, let $\{\varsigma_i\} \subset g(\varsigma(r))$. Then $\varsigma = \kappa + \overline{u_i}D_{\hat{U}}(r)$ for $\overline{u_i} \in U$. Since U is compact, there exists a subsequence $\widehat{u_{i^*}} \in \widehat{u_i}$ such that $\{\widehat{u_{i^*}}\}$ is convergent to $\overline{u} \in U$. Let $\widehat{u} = \kappa + \widehat{u}V_{\hat{U}}(r)$. Then

$$\begin{aligned}
d\left(\widehat{u_{i^*}}, \widehat{u}\right) &= \sup_{r \in I}\left(\left|\widehat{u_{i^*}} - \widehat{u}\right|\left|V_{\hat{U}}(r)\right|\right) \\
&\le \left|\widehat{u_{i^*}} - \widehat{u}\right| \sup_{r \in I}\left|V_{\hat{U}}(r)\right| \to 0, \text{ as } i^* \to +\infty.
\end{aligned}$$

Hence $g(\varsigma(r))$ is a compact subset of $\check{C}$ for all $\varsigma \in \check{C}$. Next,

$$\begin{aligned}
|V(r, \varsigma_1(r)) - V(r, \varsigma_2(r))| &\le \int_{r_0}^{r} |V(t, \varsigma_1(t)) - V(t, \varsigma_2(t))|\, dt \\
&\le \frac{\alpha_2}{\alpha_1} \int_{r_0}^{r} |\varsigma_1(t) - \varsigma_2(t)|^h\, dt \\
&\le \frac{\alpha_2}{\alpha_1} \sup_{r \in I} |\varsigma_1(t) - \varsigma_2(t)|^h \int_{r_0}^{r} dt \\
&= \frac{\alpha_2}{\alpha_1} |r - r_0| \left[d_b(\varsigma_1, \varsigma_2)\right]^h \\
&\le \frac{1}{\alpha_1}\left(\frac{\alpha_1}{\alpha_2}\right)^{h-2} \left[d_b(\varsigma_1, \varsigma_2)\right]^h.
\end{aligned}$$

Therefore, we get

$$\max_{\overline{a}\in UV_{\hat{U}}(r)} d_b\left(\overline{a}, UV_y(r)\right) \leq \left(\frac{\alpha_1}{\alpha_2}\right)^{h-2} \left[d_b(\varsigma_1,\varsigma_2)\right]^h.$$

Similarly,

$$\max_{\overline{b}\in UV_y(r)} d_b\left(\overline{b}, UV_{\hat{U}}(r)\right) \leq \left(\frac{\alpha_1}{\alpha_2}\right)^{h-2} \left[d_b\left(\widehat{\ell_1},\widehat{\ell_2}\right)\right]^h.$$

Hence (4.2) implies that

$$H_b\left[d_b(g(\varpi_1), g(\varpi_2))\right] \leq \left(\frac{\alpha_1}{\alpha_2}\right)^{h-2} \left[\check{d}_b(\varsigma_1,\varsigma_2)\right]^h.$$

Taking $\varphi(\varsigma)=\varsigma$, $\varsigma>0$ and $\xi(\varsigma)=\left(\frac{\alpha_1}{\alpha_2}\right)^{h-2}\varsigma^h$, $\varsigma\in E$ with $d_b(\varsigma_1,\varsigma_2)<\frac{\alpha_2}{\alpha_1}$, we get

$$\varphi[H_b d_b(g(\varpi_1), g(\varpi_2))] \leq \varphi\left[\xi\left(d_b(\varpi_1,\varpi_2)\right)\right] \text{ for all } \varpi_1,\varpi_2\in\check{C} \text{ with } d_b(\varsigma_1,\varsigma_2)\in E.$$

Hence the requied conditions (i)-(ii) are equivalent to (a)-(b) of Corollary 2.3. So there exists a fixed point $c^*(\in\Lambda)$ in $\check{C}$, which is a bounded solution of (4.1). □

5. Conclusions

The paper deals with the pre-existing results of fixed point for multi-valued maps satisfying φ-contraction via b-B-GGF in the context of b-metric space. Within this frame work, we introduced two related fixed point results in b-metric space. Afterwards, the results have been explained by rendering concrete examples and some foremost corollaries have been deduced from the main results. At the end, we have proved existence theorem for the system of multi-valued integral inclusion.

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Conflict of interest

The authors declare that they have no competing interests.

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