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NECESSARY CONDITIONS FOR DOUBLE SUMMABILITY FACTOR THEOREM

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Abstract. We obtain necessary conditions for the series $\sum\sum c_{mn}$, which is absolutely summable of order k by a doubly triangular matrix method A, to be such that $\sum\sum c_{mn}\lambda_{mn}$ is absolutely summable of order k by a doubly triangular matrix B.

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1. Introduction

A doubly infinite matrix $A = (a_{mnij})$ is said to be doubly triangular if $a_{mnij} = 0$ for $i > m$ and $j > n$. The mn-th terms of the A-transform of a double sequence $\{s_{mn}\}$ is defined by

$$T_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{mnij} s_{ij}.$$

A series $\sum\sum c_{mn}$, with partial sums s_{mn} is said to be absolutely A-summable, of order $k \geq 1$, if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|\Delta_{11}T_{m-1,n-1}|^k < \infty, \tag{1.1}$$

where for any double sequence $\{u_{mn}\}$, and for any four-fold sequence $\{a_{mnij}\}$, we define

$$\begin{aligned}
\Delta_{11}u_{mn} &= u_{mn} - u_{m+1,n} - u_{m,n+1} + u_{m+1,n+1},\\
\Delta_{11}a_{mnij} &= a_{mnij} - a_{m+1,n,i,j} - a_{m,n+1,i,j} + a_{m+1,n+1,i,j},\\
\Delta_{ij}a_{mnij} &= a_{mnij} - a_{m,n,i+1,j} - a_{m,n,i,j+1} + a_{m,n,i+1,j+1},\\
\Delta_{i0}a_{mnij} &= a_{mnij} - a_{m,n,i+1,j}, \quad \text{and}\\
\Delta_{0j}a_{mnij} &= a_{mnij} - a_{m,n,i,j+1}.
\end{aligned}$$

The one-dimensional version of (1.1) appears in [1].

Associated with A there are two matrices $\overline{A}$ and $\hat{A}$ defined by

$$\bar{a}_{mnij} = \sum_{\mu=i}^{m} \sum_{\nu=j}^{n} a_{mn\mu\nu},\, 0 \le i \le m,\, 0 \le j \le n,\, m,n = 0,1,\ldots,$$

and

$$\hat{a}_{mnij} = \Delta_{11}\bar{a}_{m-1,n-1,i,j},\, 0 \le i \le m,\, 0 \le j \le n,\, m,n = 1,2,\ldots.$$

It is easily verified that $\hat{a}_{0000} = \bar{a}_{0000} = a_{0000}$. In [3] it is shown that

$$\hat{a}_{mnij} = \sum_{\mu=0}^{i-1} \sum_{\nu=0}^{j-1} \Delta_{11} a_{m-1,n-1,\mu,\nu}.$$

Thus $\hat{a}_{mni0} = \hat{a}_{mn0j} = 0$.

Let x_{mn} denote the mn-th term of the A-transform of the sequence of partial sums $\{s_{mn}\}$ of the series $\sum\sum c_{mn}$.

Then

$$\begin{aligned} x_{mn} &= \sum_{i=0}^{m} \sum_{j=0}^{n} a_{mnij} s_{ij} = \sum_{i=0}^{m} \sum_{j=0}^{n} a_{mnij} \sum_{\mu=0}^{i} \sum_{\nu=0}^{j} c_{\mu\nu}\lambda_{\mu\nu} \\ &= \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \sum_{i=\mu}^{m} \sum_{j=\nu}^{n} a_{mnuv} c_{\mu\nu}\lambda_{\mu\nu} \\ &= \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} \bar{a}_{mn\mu\nu} c_{\mu\nu}\lambda_{\mu\nu}, \end{aligned}$$

and a direct calculation verifies that

$$X_{mn} := \Delta_{11} x_{m-1,n-1} = \sum_{\nu=1}^{m} \sum_{\mu=1}^{n} \hat{a}_{mn\mu\nu} c_{\mu\nu}\lambda_{\mu\nu},$$

since

$$\bar{a}_{m-1,n-1,m,\nu} = a_{m-1,n-1,\mu,n} = \hat{a}_{m,n-1,\mu,n} = \hat{a}_{m-1,n,m,n} = 0.$$

In a recent paper Savas and Rhoades[2] obtained sufficient conditions for the series $\sum\sum c_{mn}$, which is absolutely summable of order k by a doubly triangular matrix method A, to be such that $\sum\sum c_{mn}\lambda_{mn}$ is absolutely summable of order k by a doubly triangular matrix B.

In this paper we obtain necessary conditions for the series $\sum\sum c_{mn}$, which is absolutely summable of order k by a doubly triangular matrix method A, to be such that $\sum\sum c_{mn}\lambda_{mn}$ is absolutely summable of order k by a doubly triangular matrix B.

2. MAIN THEOREM

Theorem 1. *Let A and B be doubly triangular matrices with A satisfying*

$$\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\Delta_{uv}\hat{a}_{mnuv}|^k = O(M^k(\hat{a}_{uvuv})), \tag{2.1}$$

where

$$M(\hat{a}_{uvuv}) := \max\{|\hat{a}_{uvuv}|, |\Delta_{u0}\hat{a}_{u+1v,u,v}|, |\Delta_{0v}\hat{a}_{uv+1,u,v}|\}.$$

Then the necessary conditions of the fact that the $|A|_k$ summability of $\sum\sum c_{mn}$ implies the $|B|_k$ summability of $\sum\sum c_{mn}\lambda_{mn}$ are the following items:

(i) $|\hat{b}_{uvuv}\lambda_{uv}| = O(M(\hat{a}_{uvuv}))$,

(ii) $|\Delta_{u0}\hat{b}_{u+1,v,u,v}\lambda_{uv}| = O(M(\hat{a}_{uvuv}))$,

(iii) $|\Delta_{0v}\hat{b}_{u,v+1,u,v}\lambda_{uv}| = O(M(\hat{a}_{uvuv}))$,

(iv) $\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\Delta_{uv}\hat{b}_{mnuv}\lambda_{uv}|^k = O((uv)^{k-1}M^k(\hat{a}_{uvuv}))$,

(v) $\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\hat{b}_{m,n,u+1,v+1}\lambda_{u+1,v+1}|^k = O\Big(\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\hat{a}_{m,n,u+1,v+1}|^k\Big)$.

Proof. For $k \geq 1$ define

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|Y_{mn}|^k < \infty, \tag{2.2}$$

whenever

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|X_{mn}|^k < \infty, \tag{2.3}$$

where

$$Y_{mn} = \Delta_{11}y_{m-1,n-1},$$

$$y_{mn} = \sum_{i=0}^{m}\sum_{j=0}^{n}\bar{b}_{mnij}c_{ij}\lambda_{ij}.$$

The space of sequences satisfying (2.3) is a Banach space if it is normed by

$$\|X\| = \Big(|X_{00}|^k + |X_{01}|^k + |X_{10}|^k + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|X_{mn}|^k\Big)^{1/k}. \tag{2.4}$$

We shall also consider the space of sequences $\{Y_{mn}\}$ that satisfy (2.2). This space is also a BK-space with respect to the norm

$$\|Y\| = \left(|Y_{00}|^k + |Y_{01}|^k + |Y_{10}|^k + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{k-1}|Y_{mn}|^k\right)^{1/k}. \tag{2.5}$$

The transformation $\sum_{i=0}^{m}\sum_{j=0}^{n}\bar{b}_{mnij}c_{ij}\lambda_{ij}$ maps sequences satisfying (2.3) into sequences satisfying (2.2). By the Banach-Steinhaus Theorem there exists a constant $K > 0$ such that

$$\|Y\| \le K\|X\|. \tag{2.6}$$

For fixed u, v, the sequence $\{c_{ij}\}$ is defined by $c_{uv} = c_{u+1,v+1} = 1, c_{u+1,v} = c_{u,v+1} = -1, c_{ij} = 0$, otherwise,

$$X_{mn} = \begin{cases} 0, & m \le u, n < v, \\ 0, & m < u, n \le v, \\ \hat{a}_{mnuv}, & m = u, n = v, \\ \Delta_{u0}\hat{a}_{mnuv}, & m = u+1, n = v, \\ \Delta_{0v}\hat{a}_{mnuv}, & m = u, n = v+1, \\ \Delta_{uv}\hat{a}_{mnuv}, & m > u, n > v, \end{cases}$$

and

$$Y_{mn} = \begin{cases} 0, & m \le u, n < v, \\ 0, & m < u, n \le v, \\ \hat{b}_{mnuv}\lambda_{uv}, & m = u, n = v, \\ \Delta_{u0}\hat{b}_{mnuv}\lambda_{uv}, & m = u+1, n = v, \\ \Delta_{0v}\hat{b}_{mnuv}\lambda_{uv}, & m = u, n = v+1, \\ \Delta_{uv}\hat{b}_{mnuv}\lambda_{uv}, & m > u, n > v. \end{cases}$$

From (2.4) and (2.5) it follows that

$$\begin{aligned} \|X\| = \Big\{&(uv)^{k-1}|\hat{a}_{uvuv}|^k + ((u+1)v)^{k-1}|\Delta_{u0}a_{u+1,v,u,v}|^k \\ &+ (u(v+1))^{k-1}|\Delta_{0v}a_{u,v+1u,v}|^k \\ &+ \sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\Delta_{uv}\hat{a}_{mnuv}|^k\Big\}^{1/k}, \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \|Y\| = \Big\{&(uv)^{k-1}\hat{b}_{uvuv}\lambda_{uv}|^k + ((u+1)v)^{k-1}|\Delta_{u0}\hat{b}_{u+1,v,u,v}\lambda_{uv}|^k \\ &+ (u(v+1))^{k-1}|\Delta_{0v}\hat{b}_{u,v+1u,v}\lambda_{uv}|^k \end{aligned} \tag{2.8}$$

$$+\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\Delta_{uv}\hat{a}_{mnuv}\lambda_{uv}|^k\Big\}^{1/k}.$$

Substituting (2.7) and (2.8) into (2.6), along with (2.1), gives

$$(uv)^{k-1}|\hat{b}_{uvuv}\lambda_{uv}|^k+((u+1)v)^{k-1}|\Delta_{u0}\hat{b}_{u+1v,u,v}\lambda_{uv}|^k$$
$$+(u(v+1))^{k-1}|\Delta_{0v}\hat{b}_{u,v+1u,v}\lambda_{uv}|^k$$
$$+\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\Delta_{uv}\hat{b}_{mnuv}\lambda_{uv}|^k\leq K^k\Big\{(uv)^{k-1}|\hat{a}_{uvuv}|^k$$
$$((u+1)v)^{k-1}|\Delta_{u0}\hat{a}_{u+1v,u,v}|^k+(u(v+1))^{k-1}\Delta_{u0}\hat{a}_{u,v+1u,v}|^k$$
$$+\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\Delta_{uv}\hat{a}_{mnuv}|^k\Big\}$$
$$=K^k\{O(1)(uv)^{k-1}M^k(\hat{a}_{uvuv})\}.$$

The above inequality implies that each term of the left hand side is $O(\{(uv)^{k-1}M^k(\hat{a}_{uvuv})\})$.

Using the first term one obtains

$$(uv)^{k-1}|\hat{b}_{uvuv}\lambda_{uv}|^k=O(\{(uv)^{k-1}M^k(\hat{a}_{uvuv})\}).$$

Thus

$$|\hat{b}_{uvuv}\lambda_{uv}|=O(M(\hat{a}_{uvuv})),$$

which is condition (i).

In a similar manner one obtains conditions (ii) - (iv).

Using the sequence, defined by $c_{u+1,v+1}=1$, and $c_{ij}=0$ otherwise, yields

$$X_{mn}=\begin{cases}0, & m\leq u+1, n\leq v,\\ 0, & m\leq u, n\leq v+1,\\ \hat{a}_{m,n,u+1,v+1}, & m\geq u+1, n\geq v+1\end{cases}$$

and

$$Y_{mn}=\begin{cases}0. & m\leq u+1, n\leq v,\\ 0, & m\leq u, n\leq v+1,\\ \hat{b}_{m,n,u+1,v+1}\lambda_{u+1,v+1}, & m\geq u+1, n\geq v+1.\end{cases}$$

The corresponding norms are

$$\|X\|=\Big\{\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\hat{a}_{m,n,u+1,v+1}|^k\Big\}^{1/k}$$

and

$$\|Y\| = \left\{ \sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\hat{b}_{m,n,u+1,v+1} \lambda_{u+1,v+1}|^k \right\}^{1/k}.$$

Applying (2.6), we have

$$\sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\hat{b}_{m,n,u+1,v+1} \lambda_{u+1,v+1}|^k$$
$$\leq K^k \left\{ \sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\hat{a}_{m,n,u+1,v+1}|^k \right\},$$

which implies (v). □

Every summability factor theorem becomes an inclusion theorem by setting each $\lambda_{mn} = 1$.

Corollary 1. *Let A and B two doubly triangular matrices, A satisfying (2.1). Then necessary conditions of the fact that the $|A|_k$ summability of $\sum\sum c_{mn}$ implies the $|B|_k$ summability of $\sum\sum c_{mn}$ are the following items:*

(i) $|\hat{b}_{uvuv}| = O(M(\hat{a}_{uvuv}))$,

(ii) $|\Delta_{u0} \hat{b}_{u+1,v,u,v}| = O(M(\hat{a}_{uvuv}))$,

(iii) $|\Delta_{0v} \hat{b}_{u,v+1,u,v}| = O(M(\hat{a}_{uvuv}))$,

(iv) $\displaystyle\sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\Delta_{uv} \hat{b}_{mnuv}|^k = O((uv)^{k-1} M^k(\hat{a}_{uvuv}))$, *and*

(v) $\displaystyle\sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\hat{b}_{m,n,u+1,v+1}|^k$

$$= O\left(\left\{ \sum_{m=u+1}^{\infty} \sum_{n=v+1}^{\infty} (mn)^{k-1} |\hat{a}_{m,n,u+1,v+1}|^k \right\}^{1/k} \right).$$

Proof. To prove Corollary 1 simply put $\lambda_{mn} = 1$ in Theorem 1. □

We shall call a doubly infinite matrix a product matrix if it can be written as the termwise product of two singly infinite matrices F and G; i.e., $a_{mnij} = f_{mi} g_{nj}$ for each i, j, m, n.

A doubly infinite weighted mean matrix P has nonzero entries p_{ij}/P_{mn}, where p_{00} is positive and all of the other p_{ij} are nonnegative, and $P_{mn} := \sum_{i=0}^{m} \sum_{j=0}^{n} p_{ij}$. If P is a product matrix then the nonzero entries are $p_i q_j / P_m Q_n$, where $p_0 > 0, p_i > 0$ for $i > 0, q_0 > 0, q_i \geq 0$ for $j > 0$ and $P_m := \sum_{i=0}^{m} p_i, Q_n := \sum_{j=0}^{n} q_j$. Now we have the following corollary

Corollary 2. *Let B be a doubly triangular matrix, P a product weighted mean matrix satisfying*

$$\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}\left|\Delta_{uv}\Big(\frac{p_m q_n P_{u-1}Q_{v-1}}{P_m P_{m-1}Q_n Q_{n-1}}\Big)\right|^k = O\Big(\frac{p_u q_v}{P_u Q_v}\Big).$$

Then necessary conditions for $\sum\sum c_{mn}$ summable $|P|_k$ to imply that $\sum\sum c_{mn}\lambda_{mn}$ is summable $|B|_k$ are

(i) $|\hat{b}_{uvuv}\lambda_{uv}| = O\Big(\frac{p_u q_v}{P_u Q_v}\Big)$,

(ii) $|\Delta_{u0}\hat{b}_{u+1,v,u,v}\lambda_{uv}| = O\Big(\frac{p_u q_v}{P_u Q_v}\Big)$,

(iii) $|\Delta_{0v}\hat{b}_{u,v+1,u,v}\lambda_{uv}| = O\Big(\frac{p_u q_v}{P_u Q_v}\Big)$,

(iv) $\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\Delta_{uv}\hat{b}_{uvuv}\lambda_{uv}|^k = O\Big((uv)^{k-1}\Big(\frac{p_u q_v}{P u Q_v}\Big)^k\Big)$, *and*

(v) $\sum_{m=u+1}^{\infty}\sum_{n=v+1}^{\infty}(mn)^{k-1}|\hat{b}_{m,n,u+1,v+1}\lambda_{u+1,v+1}|^k = O(1)$.

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References

[1] B. E. Rhoades and E. Savaş, "General summability factor theorems and applications," *Sarajevo J. Math.*, vol. 1(13), no. 1, pp. 59–73, 2005.

[2] E. Savaş and B. E. Rhoades, "Double summability factor theorems and applications," *Math. Inequal. Appl.*, vol. 10, no. 1, pp. 125–149, 2007.

[3] E. Savaş and B. E. Rhoades, "Double absolute summability factor theorems and applications," *Nonlinear Anal.*, vol. 69, no. 1, pp. 189–200, 2008.

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