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Research Article

Summability of Double Independent Random Variables

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We will examine double sequence to double sequence transformation of independent identically distribution random variables with respect to four-dimensional summability matrix methods. The main goal of this paper is the presentation of the following theorem. If $\max_{k,l}|a_{m,n,k,l}| = \max_{k,l}|a_{m,k}a_{n,l}| = O(m^{-\gamma_1})O(n^{-\gamma_2})$, $\gamma_1,\gamma_2 > 0$, then $E|\check{X}|^{1+1/\gamma_1} < \infty$ and $E|\check{\check{X}}|^{1+1/\gamma_2} < \infty$ imply that $Y_{m,n} \to \mu$ almost sure P-convergence.

1. Introduction

Let $[X_{k,l}]$ be a factorable double sequence of independent, identically distributed random variables with $E|X_{k,l}| < \infty$ and $E(X_{k,l}) = \mu$. Let $A = a_{m,n,k,l}$ be a factorable double sequence to double sequence transformation defined as

$$(Ax)_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l}x_{k,l}. \tag{1.1}$$

These factorable sequences and matrices will be used to characterize such transformations with respect to Robison and Hamilton-type conditions (see [1, 2]). That is,regularity conditions of the following type. The four-dimensional matrix A is RH-regular if and only if

RH_1: P-$\lim_{m,n} a_{m,n,k,l} = 0$ for each k and l;

RH_2: P-$\lim_{m,n}\sum_{k,l} a_{m,n,k,l} = 1$;

RH_3: P-$\lim_{m,n}\sum_{k}|a_{m,n,k,l}| = 0$ for each l;

RH_4: P-$\lim_{m,n}\sum_{l}|a_{m,n,k,l}| = 0$ for each k;

RH_5: $\sum_{k,l}|a_{m,n,k,l}|$ is P-convergent; and

RH_6: there exist positive numbers A and B such that $\sum_{k,l>B}|a_{m,n,k,l}| < A$.

Throughout this paper, we will denote $\sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} X_{k,l}$ by $Y_{m,n}$ and examine $Y_{m,n}$ with respect to the Pringsheim converges. To accomplish this goal, we begin by presenting and prove the following theorem. A necessary and sufficient condition that $Y_{m,n} = \breve{Y}_m \breve{\breve{Y}}_n$ P-converges to μ in probability is that $\max_{k,l}|a_{m,n,k,l}| = \max_{k,l}|a_{m,k} a_{n,l}|$ converges to 0 in the Pringsheim sense. This theorem and other similar to it will be used in the pursuit of establishing the following. If $\max_{k,l}|a_{m,n,k,l}| = \max_{k,l}|a_{m,k} a_{n,l}| = O(m^{-\gamma_1}) O(n^{-\gamma_2})$, $\gamma_1, \gamma_2 > 0$, then

$$E|\breve{X}|^{1+1/\gamma_1} < \infty, \qquad E|\breve{\breve{X}}|^{1+1/\gamma_2} < \infty \tag{1.2}$$

implies that $Y_{m,n} \to \mu$ almost sure P-convergence.

2. Definitions, notations, and preliminary results

Let us begin by presenting Pringsheim's notions of convergence and divergence of double sequences.

Definition 2.1 (see [3]). A double sequence $x = [x_{k,l}]$ has *Pringsheim limit* L (denoted by P-$\lim x = L$) provided that given $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. We will describe such an x more briefly as "P-convergent."

Definition 2.2. A double sequence x is called *definite divergent*, if for every (arbitrarily large) $G > 0$ there exist two natural numbers n_1 and n_2 such that $|x_{n,k}| > G$ for $n \geq n_1$, $k \geq n_2$.

Throughout this paper, we will also denote $\sum_{k,l=1,1}^{\infty,\infty}$ by $\sum_{k,l}$. Using these definitions, Robison and Hamilton presented a series of concepts and matrix characterization of P-convergence. The first definition they both presented was the following. The four-dimensional matrix A is said to be RH-regular if it maps every bounded P-convergent sequence into a P-convergent sequence with the same P-limit. The assumption of boundedness was made because a double sequence which is P-convergent is not necessarily bounded. They both independently presented the following Silverman-Toeplitz type characterization of RH-regularity [4, 5].

Theorem 2.3. *The four-dimensional matrix A is RH-regular if and only if*

RH_1: *P*-$\lim_{m,n} a_{m,n,k,l} = 0$ *for each k and l;*

RH_2: *P*-$\lim_{m,n} \sum_{k,l} a_{m,n,k,l} = 1$;

RH_3: *P*-$\lim_{m,n} \sum_{k} |a_{m,n,k,l}| = 0$ *for each l;*

RH_4: *P*-$\lim_{m,n} \sum_{l} |a_{m,n,k,l}| = 0$ *for each k;*

RH_5: $\sum_{k,l} |a_{m,n,k,l}|$ *is P-convergent; and*

RH_6: *there exist positive numbers A and B such that* $\sum_{k,l>B} |a_{m,n,k,l}| < A$.

Following Robison and Hamilton work, Patterson in [6] presented the following two notions of subsequence of a double sequence.

Definition 2.4. The double sequence $[y]$ is a double *subsequence* of the sequence $[x]$ provided that there exist two increasing double index sequences $\{n_j\}$ and $\{k_j\}$ such that if $z_j = x_{n_j,k_j}$, then y is formed by

$$\begin{array}{cccc} z_1 & z_2 & z_5 & z_{10} \\ z_4 & z_3 & z_6 & — \\ z_9 & z_8 & z_7 & — \\ — & — & — & —. \end{array} \tag{2.1}$$

Definition 2.5 (Patterson [6]). A number β is called a *Pringsheim limit point* of the double sequence $[x]$ provided that there exists a subsequence $[y]$ of $[x]$ that has Pringsheim limit β: P-lim$[y] = \beta$.

Using these definitions, Patterson presented a series of four-dimensional matrix characterizations of such sequence spaces. Let $\{x_{k,l}\}$ be a double sequence of real numbers and, for each n, let $\alpha_n = \sup_n\{x_{k,l} : k, l \geq n\}$. Patterson [7] also extended the above notions with the presentation of the following. The *Pringsheim limit superior* of $[x]$ is defined as follows:

(1) if $\alpha = +\infty$ for each n, then P-lim sup $[x] := +\infty$;

(2) if $\alpha < \infty$ for some n, then P-lim sup $[x] := \inf_n\{\alpha_n\}$.

Similarly, let $\beta_n = \inf_n\{x_{k,l} : k, l \geq n\}$. Then the *Pringsheim limit inferior* of $[x]$ is defined as follows:

(1) if $\beta_n = -\infty$ for each n, then P-lim inf $[x] := -\infty$;

(2) if $\beta_n > -\infty$ for some n, then P-lim inf $[x] := \sup_n\{\beta_n\}$.

3. Main result

The analysis of double sequences of random variables via four-dimensional matrix transformations begins with the following theorem. However, it should be noted that the relationship between our main theorem that is stated above and the next four theorems will be apparent in their statements and proofs.

Theorem 3.1. *A necessary and sufficient condition that* $Y_{m,n} = \breve{Y}_m \breve{\breve{Y}}_n$ *P-converges to* μ *in probability is that* $\max_{k,l}|a_{m,n,k,l}| = \max_{k,l}|a_{m,k}a_{n,l}|$ *converges to* 0 *in the Pringsheim sense.*

Proof. First, note that

$$\lim_{\breve{T}\to\infty} \breve{T}P[|\breve{X}| \geq \breve{T}] = 0, \qquad \lim_{\breve{\breve{T}}\to\infty} \breve{\breve{T}}P[|\breve{\breve{X}}| \geq \breve{\breve{T}}] = 0 \tag{3.1}$$

because $E|\breve{X}| < \infty$ and $E|\breve{\breve{X}}| < \infty$. Let $T = \breve{T}\breve{\breve{T}}$, $X_{m,n,k,l} = \breve{X}_{m,k}\breve{\breve{X}}_{n,l}$, $a_{m,n,k,l}X_{k,l} = a_{m,k}\breve{X}_k a_{n,l}\breve{\breve{X}}_l$, and $Z_{m,n} = \breve{Z}_m\breve{\breve{Z}}_n = \sum_{k,l} X_{m,n,k,l}$. For sufficiently large m and n and since $\max_{k,l}|a_{m,n,k,l}|$ is a P-null sequence, it follows from (3.1) that

$$\begin{aligned} P[Z_{m,n} \neq Y_{m,n}] &\leq \sum_{k,l} P[X_{m,n,k,l} \neq a_{m,n,k,l} X_{k,l}] \\ &= \sum_{k,l} P\left[|\breve{X}| \geq \frac{1}{|a_{m,k}|}; |\breve{\breve{X}}| \geq \frac{1}{|a_{n,l}|} \right] \\ &\leq \epsilon \sum_{k,l} |a_{m,n,k,l}| \\ &\leq \epsilon M, \end{aligned} \tag{3.2}$$

where M is define by RH_6 of regularity conditions. Therefore, it suffices to show that

$$\text{P-}\lim_{m,n} Z_{m,n} = \mu \text{ in probability.} \tag{3.3}$$

Observe that

$$E(Z_{m,n}) - \mu = \sum_{k,l} a_{m,n,k,l} \left(\int_{|\breve{x}|<1/|a_{m,k}|} \breve{x}\, d\breve{F} \int_{|\breve{\breve{x}}|<1/|a_{n,l}|} \breve{\breve{x}}\, d\breve{\breve{F}} - \mu \right) + \mu \left(\sum_{k,l} a_{m,n,k,l} - 1 \right), \tag{3.4}$$

which is a P-null sequence. Since

$$\begin{aligned} \frac{1}{\breve{T}\breve{\breve{T}}} \int_{|\breve{x}|<\breve{T}} \int_{|\breve{\breve{x}}|<\breve{\breve{T}}} \breve{x}^2 \breve{\breve{x}}^2\, d\breve{F}\, d\breve{\breve{F}} &= \frac{1}{\breve{T}\breve{\breve{T}}} \{ -\breve{T}^2 P[|\breve{X}| \geq \breve{T}] \cdot (-\breve{\breve{T}}^2 P[|\breve{\breve{X}}| \geq \breve{\breve{T}}]) \} \\ &\quad + \frac{1}{\breve{T}\breve{\breve{T}}} \left\{ 2 \int_0^{\breve{T}} \breve{x} P[|\breve{X}| \geq \breve{x}] d\breve{x} \cdot 2 \int_0^{\breve{\breve{T}}} \breve{\breve{x}} P[|\breve{\breve{X}}| \geq \breve{\breve{x}}] d\breve{\breve{x}} \right\} \end{aligned} \tag{3.5}$$

is a P-null sequence with respect to T, we have

$$\sum_{k,l} \operatorname{Var} X_{m,n,k,l} \leq \sum_{k,l} |a_{m,n,k,l}|^2 \int_{|\breve{x}|<1/|a_{m,k}|} \breve{x}^2\, d\breve{F} \int_{|\breve{\breve{x}}|<1/|a_{n,l}|} \breve{\breve{x}}^2\, d\breve{\breve{F}} \leq \epsilon \sum_{k,l} |a_{m,n,k,l}| \leq \epsilon M \tag{3.6}$$

for m and n sufficiently large, where $F = \breve{F}\breve{\breve{F}}$ and $x = \breve{x}\breve{\breve{x}}$. It is also clear that $E(\sum_{k,l} x_{m,n,k,l})^2$ is finite. Thus,

$$\sum_{k,l} \operatorname{Var} X_{m,n,k,l} = \operatorname{Var} \left(\sum_{k,l} X_{m,n,k,l} \right) \tag{3.7}$$

is finite. The result clearly follows from the Chebyshev's inequality. Thus, the sufficiency is proved.

Now, let us consider the necessary part of this theorem. Similar to Pruitt's notation [8], let $U_{k,l} = X_{k,l} - \mu$ and consider the transformation $T_{m,n} = \sum_{k,l} a_{m,n,k,l} U_{k,l}$. Our goal become showing that $T_{m,n}$ P-converges in probability to 0. Which imply that $T_{m,n}$ P-converges in law to 0. Let us consider the characteristic function of $T_{m,n,}$ that is,

$$E(e^{uT_{m,n}}) = E(e^{u\sum_{k,l} a_{m,n,k,l} U_{k,l}}) = E(\Pi_{k,l} e^{ua_{m,n,k,l} U_{k,l}}) = \Pi_{k,l} E(e^{ua_{m,n,k,l} U_{k,l}}) := \Pi_{k,l}\, g(ua_{m,n,k,l}). \tag{3.8}$$

Observe that

$$\text{P-}\lim_{m,n}\{\Pi_{k,l}g(ua_{m,n,k,l})\} = 1. \tag{3.9}$$

Because

$$|\Pi_{k,l}g(ua_{m,n,k,l})| \le |g(ua_{m,n,k,l})| \le 1 \tag{3.10}$$

for all (m,n) we have that

$$\text{P-}\lim_{m,n} g(ua_{m,n,k,l}) = 1 \tag{3.11}$$

for all (k,l). Clearly, there exists u_0 such that $|g(ua_{m,n,k,l})| < 1$ for $0 < |u| < u_0$. Let $u = u_0/2M$ then there exists a double subsequence $[a_{m,n,k_m,l_n}]$ such that

$$|ua_{m,n,k_m,l_n}| \le Mu = \frac{u_0}{2}. \tag{3.12}$$

Thus $\text{P-}\lim_{m,n}ua_{m,n,k_m,l_n} = 0$. Therefore, clearly we can choose (k_m,l_n) such that

$$|a_{m,n,k_m,l_n}| = \max_{k,l}|a_{m,n,k,l}|. \tag{3.13}$$

□

Theorem 3.2. *If* $E(|\breve{X}|)^{1+1/\gamma_1} < \infty$, $E(|\breve{\breve{X}}|)^{1+1/\gamma_2} < \infty$, *and* $\max_{k,l}|a_{m,n,k,l}| = \max_k|a_{m,k}| \cdot \max_l|a_{n,l}| \le \breve{B}m^{-\gamma_1}\breve{\breve{B}}n^{-\gamma_2}$, *then for every* $\epsilon > 0$

$$\sum_{m,n}P[|a_{m,n,k,l}X_{k,l}| \ge \epsilon \text{ for some } (k,l)] < \infty, \tag{3.14}$$

that is,

$$\sum_{m,n}P[|a_{m,k}\breve{X}_k| \ge \epsilon;\ |a_{n,l}\breve{\breve{X}}_l| \ge \epsilon \text{ for some } (k,l)] < \infty. \tag{3.15}$$

Proof. Let

$$N_{m,n}(x) = N_{m,n}(\breve{x}\breve{\breve{x}}) = \sum_{\{(k,l):1/|a_{m,k}|\le\breve{x};\,1/|a_{n,l}|\le\breve{\breve{x}}\}}|a_{m,n,k,l}|. \tag{3.16}$$

Note $x = \breve{x}\breve{\breve{x}}$, and observe that $N_{m,n}(x) = 0$ for $\breve{x} < (m)^{\gamma_1}$, $\breve{\breve{x}} < (n)^{\gamma_2}$, and $\iint_0^\infty d(N_{m,n}(x)) = \sum_{k,l}|a_{m,n,k,l}| \le M$. If

$$G(x) = P(|X| \ge x) = P(|\breve{X}| \ge \breve{x})P(|\breve{\breve{X}}| \ge \breve{\breve{x}}) = G(\breve{x})G(\breve{\breve{x}}), \tag{3.17}$$

then $xG(x)$ converges to 0 in the Pringsheim sense because $E(X) < \infty$ and recalled that $T = \breve{T}\breve{\breve{T}}$. Therefore,

$$\begin{aligned}
\sum_{k,l} P[|a_{m,n,k,l}x_{k,l}| \geq 1] &= \sum_{k,l} G\left(\frac{1}{|a_{m,n,k,l}|}\right) \\
&= \sum_{k,l} \frac{1}{|a_{m,n,k,l}|} G\left(\frac{1}{|a_{m,n,k,l}|}\right)|a_{m,n,k,l}| \\
&= \iint_0^{\infty} xG(x)d(N_{m,n}(x)) \\
&= N_{m,n}(T)TG(T)|_0^{\infty}|_0^{\infty} - \iint_0^{\infty} N_{m,n}(x)d(xG(x)) \\
&= \lim_{T\to\infty} N_{m,n}(T)TG(T) - \iint_0^{\infty} N_{m,n}(x)d(xG(x)) \\
&\leq M\int_{m^{\gamma_1}}^{\infty}\int_{n^{\gamma_2}}^{\infty} |d(xG(x))| \\
&= M\int_{m^{\gamma_1}}^{\infty}\int_{n^{\gamma_2}}^{\infty} |d(\breve{x}G(\breve{x}))d(\breve{\breve{x}}G(\breve{\breve{x}}))|.
\end{aligned} \tag{3.18}$$

Our goal now is to get an estimate for $\int_{m^{\gamma_1}}^{\infty}\int_{n^{\gamma_2}}^{\infty}|d(xG(x))|$. To this end observe that, for $z < y$

$$yG(y) - zG(z) = (y - z)G(z) + y(G(z) - G(y)), \tag{3.19}$$

where $(y - z)G(z)$ and $y(G(z) - G(y))$ are increasing and decreasing functions of y, respectively. Thus

$$\int_{\breve{z}}^{\breve{y}}\int_{\breve{\breve{z}}}^{\breve{\breve{y}}} d|xG(x)| \leq [(\breve{y} - \breve{z})G(\breve{z}) + \breve{y}(G(\breve{z}) - G(\breve{y}))] \cdot [(\breve{\breve{y}} - \breve{\breve{z}})G(\breve{\breve{z}}) + \breve{\breve{y}}(G(\breve{\breve{z}}) - G(\breve{\breve{Y}}))]. \tag{3.20}$$

The last inequality grant us the following:

$$\begin{aligned}
&\int_{m^{\gamma_1}}^{\infty}\int_{n^{\gamma_2}}^{\infty} |d(\breve{x}G(\breve{x}))d(\breve{\breve{x}}G(\breve{\breve{x}}))| \\
&\quad = \sum_{i,j=m,n}^{\infty,\infty} \int_{i^{\gamma_1}}^{(i+1)^{\gamma_1}} \int_{j^{\gamma_2}}^{(j+1)^{\gamma_2}} |d(\breve{x}G(\breve{x}))d(\breve{\breve{x}}G(\breve{\breve{x}}))| \\
&\quad \leq \sum_{i,j=m,n}^{\infty,\infty} \{[(i+1)^{\gamma_1} - i^{\gamma_1}]G(i^{\gamma_1}) \cdot [(j+1)^{\gamma_2} - j^{\gamma_2}]G(j^{\gamma_2})\} \\
&\qquad + \sum_{i,j=m,n}^{\infty,\infty} \{(i+1)^{\gamma_1}[G(i^{\gamma_1}) - G((i+1)^{\gamma_1})] \cdot (j+1)^{\gamma_2}[G(j^{\gamma_2}) - G((j+1)^{\gamma_2})]\}.
\end{aligned} \tag{3.21}$$

Therefore,

$$\int_{m^{\gamma_1}}^{\infty}\int_{n^{\gamma_2}}^{\infty}|d(\breve{x}G(\breve{x}))d(\breve{\breve{x}}G(\breve{\breve{x}}))| \leq 2\sum_{i,j=m,n}^{\infty,\infty}\{(i+1)^{\gamma_1}[G(i^{\gamma_1})-G((i+1)^{\gamma_1})]\cdot(j+1)^{\gamma_2}[G(j^{\gamma_2})-G((j+1)^{\gamma_2})]\}.$$

$$\begin{aligned}
&\sum_{m,n}^{\infty}P[|a_{m,n,k,l}X_{k,l}|\geq\epsilon \text{ for some } (k,l)]\\
&\quad\leq\sum_{m,n}^{\infty}\sum_{k,l}^{\infty}P[|a_{m,n,k,l}X_{k,l}|\geq\epsilon]\\
&\quad\leq 2M\sum_{m,n=1,1}^{\infty,\infty}\sum_{i,j=m,n}^{\infty,\infty}\{(i+1)^{\gamma_1}[G(i^{\gamma_1})-G((i+1)^{\gamma_1})]\cdot(j+1)^{\gamma_2}[G(j^{\gamma_2})-G((j+1)^{\gamma_2})]\}\\
&\quad=2M\sum_{i,j=1,1}^{\infty,\infty}\{(i+1)^{\gamma_1}[G(i^{\gamma_1})-G((i+1)^{\gamma_1})]\cdot(j+1)^{\gamma_2}[G(j^{\gamma_2})-G((j+1)^{\gamma_2})]\}\\
&\quad\leq 2^{1+\gamma_1}2^{1+\gamma_2}M\int\int|\breve{x}|^{1+1/\gamma_1}|\breve{\breve{x}}|^{1+1/\gamma_2}d\breve{F}(\breve{x})d\breve{\breve{F}}(\breve{\breve{x}})\\
&\quad<\infty.
\end{aligned} \tag{3.22}$$

□

Theorem 3.3. *Let x and F be define as in Theorem 3.2. If $E|\breve{X}|^{1+1/\gamma_1}<\infty$, $E|\breve{\breve{X}}|^{1+1/\gamma_2}<\infty$, and* $\max_{k,l}|a_{m,n,k,l}|=\max_k|a_{m,k}|\cdot\max_l|a_{n,l}|\leq\breve{B}m^{-\gamma_1}\breve{\breve{B}}n^{-\gamma_2}$ *then for $\alpha_1<\gamma_1/2(\gamma_1+1)$ and $\alpha_2<\gamma_2/2(\gamma_2+1)$*

$$\sum_{m,n}P[|a_{m,n,k,l}X_{k,l}|\geq m^{\alpha_1}n^{\alpha_2} \textit{ for at least two pairs } (k,l)]<\infty, \tag{3.23}$$

that is,

$$\sum_{m,n}P[|a_{m,k}\breve{X}_k|\geq m^{\alpha_1};\ |a_{n,l}\breve{\breve{X}}_l|\geq n^{\alpha_2} \textit{ for at least two pairs } (k,l)]<\infty. \tag{3.24}$$

Proof. By Markov's inequality, we have the following:

$$\begin{aligned}
\sum_{m}P[|a_{m,k}\breve{X}_k|\geq m^{\alpha_1}]&\leq|a_{m,k}|^{1+1/\gamma_1}E(|\breve{x}|)^{1+1/\gamma_1}m^{\alpha_1(1+1/\gamma_1)},\\
\sum_{n}P[|a_{n,l}\breve{\breve{X}}_l|\geq n^{\alpha_2}]&\leq|a_{n,l}|^{1+1/\gamma_2}E(|\breve{\breve{x}}|)^{1+1/\gamma_2}n^{\alpha_2(1+1/\gamma_2)}.
\end{aligned} \tag{3.25}$$

Therefore,

$$\begin{aligned}
&\sum_{m,n} P[|a_{m,k}\breve{X}_k| \geq m^{\alpha_1};\ |a_{n,l}\breve{\breve{X}}_l| \geq n^{\alpha_2} \text{ for at least two pairs } (k,l)] \\
&\quad\leq \sum_{i\neq k,\, j\neq l} P[|a_{m,i}\breve{X}_i| \geq m^{\alpha_1}; |a_{m,k}\breve{X}_k| \geq m^{\alpha_1}; |a_{n,j}\breve{\breve{X}}_j| \geq n^{\alpha_2}; |a_{n,l}\breve{\breve{X}}_l| \geq n^{\alpha_2}] \\
&\quad\leq E(|\breve{x}|^{1+1/\gamma_1})^2 m^{2\alpha_1(1+1/\gamma_1)} \sum_{i\neq k} |a_{m,i}|^{1+1/\gamma_1} |a_{m,k}|^{1+1/\gamma_1} \\
&\qquad \cdot E(|\breve{\breve{x}}|^{1+1/\gamma_2})^2 n^{2\alpha_2(1+1/\gamma_2)} \sum_{j\neq l} |a_{n,j}|^{1+1/\gamma_2} |a_{n,l}|^{1+1/\gamma_2} \\
&\quad\leq E(|\breve{x}|^{1+1/\gamma_1})^2 \cdot E(|\breve{\breve{x}}|^{1+1/\gamma_2})^2 \breve{B}^{2/\gamma_1} \breve{\breve{B}}^{2/\gamma_2} M^4 m^{2[-1+\alpha_1(1+1/\gamma_1)]} n^{2[-1+\alpha_2(1+1/\gamma_2)]},
\end{aligned} \tag{3.26}$$

which is P-convergent when sum on n and m provided that $\alpha_1 < \gamma_1/2(\gamma_1+1)$ and $\alpha_2 < \gamma_2/2(\gamma_2+1)$. □

Theorem 3.4. *Let x and F be define as in Theorem 3.2. If $\mu = 0$, $E|\breve{X}|^{1+1/\gamma_1} < \infty$, $E|\breve{\breve{X}}|^{1+1/\gamma_2} < \infty$, and* $\max_{k,l}|a_{m,n,k,l}| = \max_k|a_{m,k}| \cdot \max_l|a_{n,l}| \leq \breve{B}m^{-\gamma_1}\breve{\breve{B}}n^{-\gamma_2}$ *then for $\epsilon > 0$*

$$\sum_{m,n} P\left[\widehat{\sum_{k,l}} |a_{m,n,k,l}X_{k,l}| \geq \epsilon\right] < \infty, \tag{3.27}$$

where

$$\widehat{\sum_{k,l}} a_{m,n,k,l}X_{k,l} = \sum_{\{k:|a_{m,k}X_k| < m^{-\alpha_1}\ l:|a_{n,l}X_l| < n^{-\alpha_2}\}} a_{m,n,k,l}X_{k,l}, \tag{3.28}$$

$\alpha_1 < \gamma_1$, and $\alpha_2 < \gamma_2$.

Proof. Let

$$X_{m,n,k,l} := \begin{cases} X_{m,k}; & \text{if } |a_{m,k}X_k| < m^{-\alpha_1}, \\ X_{n,l}; & \text{if } |a_{n,l}X_l| < n^{-\alpha_2}, \\ 0; & \text{otherwise,} \end{cases} \tag{3.29}$$

and $\beta_{m,n,k,l} = E(X_{m,n,k,l})$. If $a_{m,n,k,l} = 0$, then $\beta_{m,n,k,l} = \mu = 0$ and if $a_{m,n,k,l} \neq 0$, then

$$\begin{aligned}
|\beta_{m,n,k,l}| &= \left|\mu - \int_{|\breve{x}| \geq m^{-\alpha_1}|a_{m,k}|^{-1}} \int_{|\breve{\breve{x}}| \geq m^{-\alpha_2}|a_{n,l}|^{-1}} x\,dF\right| \\
&\leq \int_{|\breve{x}| \geq m^{-\alpha_1}\breve{B}^{-1} m^{\gamma_1}} \int_{|\breve{\breve{x}}| \geq n^{-\alpha_2}\breve{\breve{B}}^{-1} n^{\gamma_2}} |x|\,dF.
\end{aligned} \tag{3.30}$$

Therefore, P-$\lim_{m,n}\beta_{m,n,k,l} = 0$ uniformly in (k,l) and P-$\lim_{m,n}\sum_{k,l} a_{m,n,k,l}\beta_{m,n,k,l} = 0$. Let

$$Z_{m,n,k,l} = Z_{m,k}Z_{n,l} = X_{m,n,k,l} - \beta_{m,n,k,l}, \tag{3.31}$$

so that $E(Z_{m,n,k,l}) = 0$, $E(|Z_{m,k}|^{1+1/\gamma_1}) < c_1$, and $E(|Z_{n,l}|^{1+1/\gamma_2}) < c_2$ for some c_1 and c_2. Also $|a_{m,k}Z_{m,k}| \le 2m^{-\alpha_1}$ and $|a_{n,l}Z_{n,l}| \le 2n^{-\alpha_2}$. Observe that

$$\widehat{\sum_{k,l}} a_{m,n,k,l}X_{k,l} = \sum_{k,l} a_{m,n,k,l}X_{m,n,k,l} = \sum_{k,l} a_{m,n,k,l}Z_{m,n,k,l} + \sum_{k,l} a_{m,n,k,l}\beta_{m,n,k,l}. \tag{3.32}$$

Note for sufficiently large m and n

$$\left[\left|\widehat{\sum_{k,l}} a_{m,n,k,l}X_{k,l}\right| \ge \epsilon\right] \subset \left[\left|\sum_{k,l} a_{m,n,k,l}Z_{m,n,k,l}\right| \ge \frac{\epsilon}{2}\right]. \tag{3.33}$$

Thus it is sufficient to show that

$$\sum_{m,n} P\left[\left|\widehat{\sum_{k,l}} |a_{m,n,k,l}Z_{m,n,k,l}|\right| \ge \epsilon\right] < \infty. \tag{3.34}$$

Let η_1 and η_2 be the least integers greater than $1/\gamma_1$ and $1/\gamma_2$, respectively. Our goal now is to produce an estimate for

$$E\left(\left(\sum_k a_{m,k}Z_{m,k}\right)^{2\eta_1}\left(\sum_l a_{n,l}Z_{n,l}\right)^{2\eta_2}\right). \tag{3.35}$$

Observe that

$$E\left(\left(\sum_k a_{m,k}Z_{m,k}\right)^{2\eta_1}\left(\sum_l a_{n,l}Z_{n,l}\right)^{2\eta_2}\right) \tag{3.36}$$

is equal to

$$\sum_{k_1,k_2,\ldots,k_{2p};\ l_1,l_2,\ldots,l_{2q}} E\left(\prod_{i=1}^{2p}\prod_{j=1}^{2q} a_{m,n,k_i,l_j}Z_{m,n,k_i,l_j}\right). \tag{3.37}$$

It happens to be the case that $E((\sum_k a_{m,k}Z_{m,k})^{2\eta_1}(\sum_l a_{n,l}Z_{n,l})^{2\eta_2})$ is zero if $k_i, l_i \ne k_j, l_j$ for $i \ne j$ because the $Z_{m,n,k,l}$'s are independent and $E(Z_{m,n,k,l}) = 0$. Let us now consider the general term. Thus

$$\begin{gathered} p_1 \text{ of the } k'\text{s} = \phi_1, \ldots, p_{\theta_1} \text{ of the } k'\text{s} = \phi_{\theta_1}, \\ q_1 \text{ of the } k'\text{s} = \varphi_1, \ldots, q_{\theta_2} \text{ of the } k'\text{s} = \varphi_{\theta_2}, \\ r_1 \text{ of the } l'\text{s} = \kappa_1, \ldots, r_{\tau_1} \text{ of the } l'\text{s} = \kappa_{\tau_1}, \\ s_1 \text{ of the } l'\text{s} = \omega_1, \ldots, s_{\tau_2} \text{ of the } l'\text{s} = \omega_{\tau_2}, \end{gathered} \tag{3.38}$$

where $2 \le p_i \le 1 + 1/\gamma_1$, $q_j > 1 + 1/\gamma_1$, $2 \le r_\lambda \le 1 + 1/\gamma_2$, $s_\chi > 1 + 1/\gamma_2$,

$$\begin{gathered} \sum_{i=1}^{\theta_1} p_i + \sum_{j=1}^{\theta_2} q_j = 2\eta_1, \\ \sum_{\lambda=1}^{\tau_1} r_i + \sum_{\chi=1}^{\tau_2} s_\chi = 2\eta_2. \end{gathered} \tag{3.39}$$

Now let us consider the following expectation:

$$\begin{aligned}
&E\left(\prod_{i=1}^{\theta_1}(a_{m,\phi_i}Z_{m,\phi_i})^{p_i}\cdot\prod_{j=1}^{\theta_2}(a_{m,\varphi_j}Z_{m,\varphi_j})^{q_j}\cdot\prod_{\lambda=1}^{\tau_1}(a_{n,\kappa_\lambda}Z_{n,\kappa_\lambda})^{r_\lambda}\prod_{\chi=1}^{\tau_2}(a_{n,\omega_\chi}Z_{n,\omega_\chi})^{s_\chi}\right)\\
&\quad\le(1+c_1)^{\theta_1}(1+c_2)^{\tau_1}\cdot\prod_{\chi=1}^{\tau_2}|a_{m,\phi_i}|^{p_i}\prod_{\lambda=1}^{\tau_1}|a_{n,\kappa_\lambda}|^{r_\lambda}\\
&\qquad\cdot E\left(\prod_{j=1}^{\theta_2}(a_{m,\varphi_j}Z_{m,\varphi_j})^{q_j}\cdot\prod_{\chi=1}^{\tau_2}(a_{n,\omega_\chi}Z_{n,\omega_\chi})^{s_\chi}\right)\\
&\quad\le(1+c_1)^{\theta_1}(1+c_2)^{\tau_1}\cdot\prod_{i=1}^{\theta_1}|a_{m,\phi_i}|^{p_i}\prod_{\lambda=1}^{\tau_1}|a_{n,\kappa_\lambda}|^{r_\lambda}\\
&\qquad\cdot\prod_{j=1}^{\theta_2}|a_{m,\varphi_j}|^{1+1/\gamma_1}(2m^{-\alpha_1})^{q_j-1-1/\gamma_1}\cdot\prod_{\chi=1}^{\tau_2}|a_{n,\omega_\chi}|^{1+1/\gamma_2}(2n^{-\alpha_2})^{s_\chi-1-1/\gamma_2}\\
&\quad\le(1+c_1)^{\theta_1}(1+c_2)^{\tau_1}\cdot\prod_{i=1}^{\theta_1}|a_{m,\phi_i}||a_{m,\phi_i}|^{p_i-1}\\
&\qquad\cdot\prod_{\lambda=1}^{\tau_1}|a_{n,\kappa_\lambda}||a_{n,\kappa_\lambda}|^{r_\lambda-1}\cdot\prod_{j=1}^{\theta_2}|a_{m,\varphi_j}|^{1+1/\gamma_1}(2m^{-\alpha_1})^{q_j-1-1/\gamma_1}\\
&\qquad\cdot\prod_{\chi=1}^{\tau_2}|a_{n,\omega_\chi}|^{1+1/\gamma_2}(2n^{-\alpha_2})^{s_\chi-1-1/\gamma_2}\\
&\quad\le(1+c_1)^{\theta_1}(1+c_2)^{\tau_1}\cdot\prod_{i=1}^{\theta_1}|a_{m,\phi_i}|\prod_{\lambda=1}^{\tau_1}|a_{n,\kappa_\lambda}|\prod_{j=1}^{\theta_2}|a_{m,\varphi_j}|\prod_{\chi=1}^{\tau_2}|a_{n,\omega_\chi}|\\
&\qquad\cdot(\breve{B}m^{-\gamma_1})^{\sum_{i=1}^{\theta_1}(p_i-1)+\theta_2/\gamma_1}(2m^{-\alpha_1})^{\sum_{j=1}^{\theta_2}(q_j-1-1/\gamma_1)}\\
&\qquad\cdot(\breve{\breve{B}}n^{-\gamma_2})^{\sum_{\lambda=1}^{\tau_1}(r_\lambda-1)+\tau_2/\gamma_2}(2n^{-\alpha_2})^{\sum_{\chi=1}^{\tau_2}(s_\chi-1-1/\gamma_2)},
\end{aligned}\tag{3.40}$$

where c_1 and c_2 are upper bound for $E|Z_{m,k}|$ and $E|Z_{n,l}|$, respectively. Now let us examine the negative exponents, that is,

$$\begin{aligned}
&\gamma_1\sum_{i=1}^{\theta_1}(p_i-1)+\theta_2+\alpha_1\sum_{j=1}^{\theta_2}\left(q_j-1-\frac{1}{\gamma_1}\right),\\
&\gamma_2\sum_{\lambda=1}^{\tau_1}(r_\lambda-1)+\tau_2+\alpha_2\sum_{\chi=1}^{\tau_2}\left(s_\chi-1-\frac{1}{\gamma_2}\right).
\end{aligned}\tag{3.41}$$

Observe that, if θ_2 and τ_2 are 1 or large, then

$$\begin{aligned}
&\theta_2+\alpha_1\sum_{j=1}^{\theta_2}\left(q_j-1-\frac{1}{\gamma_1}\right)\ge1+\alpha_1\left(\eta_1-\frac{1}{\gamma_1}\right),\\
&\tau_2+\alpha_2\sum_{\chi=1}^{\tau_2}\left(s_\chi-1-\frac{1}{\gamma_2}\right)\ge1+\alpha_2\left(\eta_2-\frac{1}{\gamma_2}\right),
\end{aligned}\tag{3.42}$$

respectively. Also is $\theta_2 = \tau_2 = 0$, then

$$\gamma_1 \sum_{i=1}^{\theta_1} (p_i - 1) = \gamma_1(2\eta_1 - \theta_1) \geq \gamma_1\eta_1 \geq 1 + \gamma_1\left(\eta_1 - \frac{1}{\gamma_1}\right) \geq 1 + \alpha_1\left(\eta_1 - \frac{1}{\gamma_1}\right),$$
$$\gamma_2 \sum_{\lambda=1}^{\tau_1} (r_\lambda - 1) = \gamma_2(2\eta_2 - \tau_1) \geq \gamma_2\eta_2 \geq 1 + \gamma_2\left(\eta_2 - \frac{1}{\gamma_2}\right) \geq 1 + \alpha_2\left(\eta_2 - \frac{1}{\gamma_2}\right). \tag{3.43}$$

Thus the expected value in (3.40) is bounded by the product of

$$K_1 \prod_{i=1}^{\theta_1} |a_{m,\phi_i}| \prod_{j=1}^{\theta_2} |a_{m,\varphi_j}| m^{-1-\alpha_1(\eta_1 - 1/\gamma_1)},$$
$$K_2 \prod_{\lambda=1}^{\tau_1} |a_{n,\kappa_\lambda}| \prod_{\chi=1}^{\tau_2} |a_{n,\omega_\chi}| n^{-1-\alpha_2(\eta_2 - 1/\gamma_2)}, \tag{3.44}$$

where K_1 dependent on c_1, γ_1, $\breve{B}$; and c_2, γ_2, $\breve{\breve{B}}$, respectively.

Therefore,

$$E\left(\sum_k a_{m,k} Z_{m,k}\right)^{2\eta_1} \leq K_3 m^{-1-\alpha_1(\eta_2 - 1/\gamma_1)},$$
$$E\left(\sum_l a_{n,l} Z_{n,l}\right)^{2\eta_2} \leq K_4 n^{-1-\alpha_2(\eta_2 - 1/\gamma_2)} \tag{3.45}$$

for some K_3 and K_4 which independent on c_1, γ_1, $\breve{B}$, M and c_2, γ_2, $\breve{\breve{B}}$, M, respectively. With both independent of (m, n). Now the result follows from Markov's inequality. □

Theorem 3.5. *If* $\max_{k,l}|a_{m,n,k,l}| = \max_{k,l}|a_{m,k}a_{n,l}| = O(m^{-\gamma_1})O(n^{-\gamma_2})$, $\gamma_1, \gamma_2 > 0$, *then* $E|\breve{X}|^{1+1/\gamma_1} < \infty$ *and* $E|\breve{\breve{X}}|^{1+1/\gamma_1} < \infty$ *implies that* $Y_{m,n} \to \mu$ *almost sure P-convergence.*

Proof. Observe that

$$\sum_{k,l} a_{m,n,k,l} X_{k,l} = \sum_{k,l} a_{m,n,k,l}(X_{k,l} - \mu) + \mu \sum_{k,l} a_{m,n,k,l}. \tag{3.46}$$

Note the last term P-converge to μ because of the regularity of A. We will only consider the case when $\mu = 0$. By the Borel-Cantelli lemma, it is sufficient to prove that for $\epsilon > 0$

$$\sum_{m,n} P\left[\left|\sum_{k,l} a_{m,n,k,l} x_{k,l}\right| \geq \epsilon\right] \leq \infty. \tag{3.47}$$

At this point, the proof follows a path identical to Pruitt's proof using the above theorems and as such, the rest is omitted. □

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