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On Asymptotically Lacunary Statistical Equivalent Sequences

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Abstract : This paper presents the following definition which is a natural combination of the definition for asymptotically equivalent, statistically limit and lacunary sequences. Let θ be a lacunary sequence; the two nonnegative sequences $[x]$ and $[y]$ are said to be asymptotically lacunary statistical equivalent of multiple L provided that for every $\epsilon > 0$

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \epsilon \right\} \right| = 0$$

(denoted by $x \overset{S_\theta^L}{\sim} y$) and simply asymptotically lacunary statistical equivalent if $L = 1$. In addition, we shall also present asymptotically equivalent analogs of Fridy's and Orhan's theorems in [3].

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1 Introduction

In 1993, Marouf presented definitions for asymptotically equivalent sequences and asymptotic regular matrices. In 2003, Patterson extend these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. This paper extend the definitions presented in [5] to lacunary sequences. In addition to these definition, natural inclusion theorems shall also be presented.

2 Definitions and Notations

Definition 2.1 (Marouf, [4]) Two nonnegative sequences $[x]$, and $[y]$ are said to be *asymptotically equivalent* if

$$\lim_k \frac{x_k}{y_k} = 1$$

(denoted by $x \sim y$).

Definition 2.2 (Fridy, [2]) The sequence $[x]$ has **statistic limit** L, denoted by $st-\lim s = L$ provided that for every $\epsilon > 0$,

$$\lim_n \frac{1}{n}\Big\{\text{the number of } k \leq n : |x_k - L| \geq \epsilon\Big\} = 0.$$

The next definition is natural combination of definition (2.1) and(2.2).

Definition 2.3 (Patterson, [5]) Two nonnegative sequence $[x]$ and $[y]$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\epsilon > 0$,

$$\lim_n \frac{1}{n}\left\{\text{the number of } k < n : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\} = 0$$

(denoted by $x \overset{S_L}{\sim} y$), and simply asymptotically statistical equivalent if $L = 1$.

Following these results we introduce two new notions asymptotically lacunary statistical equivalent of multiple L and strong asymptotically lacunary equivalent of Multiple L.

By a lacunary $\theta = (k_r)$; $r = 0, 1, 2, ...$ where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1}$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and $h_r = k_r - k_{r-1}$. The ratio $\dfrac{k_r}{k_{r-1}}$ will be denoted by q_r.

Definition 2.4 Let θ be a lacunary sequence; the two nonnegative sequences $[x]$ and $[y]$ are said to be asymptotically lacunary statistical equivalent of multiple L provided that for every $\epsilon > 0$

$$\lim_r \frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| = 0$$

(denoted by $x \overset{S_\theta^L}{\sim} y$) and simply asymptotically lacunary statistical equivalent if $L = 1$. Furthermore, let S_θ^L denote the set of x and y such that $x \overset{S_\theta^L}{\sim} y$.

Definition 2.5 Let θ be a lacunary sequence; two number sequences $[x]$ and $[y]$ are strong asymptotically lacunary equivalent of multiple L provided that

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r}\left|\frac{x_k}{y_k} - L\right| = 0,$$

(denoted by $x \overset{N_\theta^L}{\sim} y$) and strong simply asymptotically lacunary equivalent if $L = 1$. In addition, let N_θ^L denote the set of x and y such that $x \overset{N_\theta^L}{\sim} y$.

3 Main Results

Theorem 3.1 *Let $\theta = \{k_r\}$ be a lacunary sequence then*

1. (a) *If $x \overset{N_\theta^L}{\sim} y$ then $x \overset{S_\theta^L}{\sim} y$*
   (b) *N_θ^L is a proper subset of S_θ^L*
2. *If $x \in l_\infty$ and $x \overset{S_\theta^L}{\sim} y$ then $x \overset{N_\theta^L}{\sim} y$*
3. *$S_\theta^L \cap l_\infty = N_\theta^L \cap l_\infty$*

where l_∞ denote the set of bounded sequences.

Proof. Part (1a): If $\epsilon > 0$ and $x \overset{N_\theta^L}{\sim} y$ then

$$\sum_{k \in I_r} \left| \frac{x_k}{y_k} - L \right| \geq \sum_{k \in I_r \& \left| \frac{x_k}{y_k} - L \right| \geq \epsilon} \left| \frac{x_k}{y_k} - L \right| \geq \epsilon \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \epsilon \right\} \right|.$$

Therefore $x \overset{S_\theta^L}{\sim} y$.

Part (1b): $N_\theta^L \subset S_\theta^L$, let $[x]$ be define as follows x_k to be $1, 2, \ldots, [\sqrt{h_r}]$ at the first $[\sqrt{h_r}]$ integers in I_r and zero otherwise. $y_k = 1$ for all k. These two satisfies the following $x \overset{S_\theta^0}{\sim} y$ but the following fails $x \overset{N_\theta^L}{\sim} y$.

Part (2): Suppose $[x]$ and $[y]$ are in l_∞ and $x \overset{S_\theta^L}{\sim} y$. Then we can assume that

$$\left| \frac{x_k}{y_k} - L \right| \leq M \quad \text{for all } k.$$

Given $\epsilon > 0$

$$\frac{1}{h_r} \sum_{k \in I_r} \left| \frac{x_k}{y_k} - L \right| = \frac{1}{h_r} \sum_{k \in I_r \& \left| \frac{x_k}{y_k} - L \right| \geq \epsilon} \left| \frac{x_k}{y_k} - L \right| + \frac{1}{h_r} \sum_{k \in I_r \& \left| \frac{x_k}{y_k} - L \right| < \epsilon} \left| \frac{x_k}{y_k} - L \right|$$
$$\leq \frac{M}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \epsilon \right\} \right| + \epsilon.$$

Therefore $x \overset{N_\theta^L}{\sim} y$.

Part (3): follows from (1) and (2). □

Theorem 3.2 *Let $\theta = \{k_r\}$ be a lacunary sequence with $\liminf q_r > 1$, then*

$$x \overset{S^L}{\sim} y \ \textit{implies}\ x \overset{S_\theta^L}{\sim} y.$$

Proof. Suppose first that $\liminf q_r > 1$, then there exists a $\delta > 0$ such that $q_r \geq 1 + \delta$ for sufficiently large r, which implies

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta}.$$

If $x \overset{S^L}{\sim} y$, then for every $\varepsilon > 0$ and for sufficiently large r, we have

$$\begin{aligned} \frac{1}{k_r}\left|\left\{k \leq k_r : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| &\geq \frac{1}{k_r}\left|\left\{k \in I_r : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| \\ &\geq \frac{\delta}{1+\delta}\frac{1}{h_r}\left|\left\{k \in I_r : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right|; \end{aligned}$$

this completes the proof. □

Theorem 3.3 *Let $\theta = \{k_r\}$ be a lacunary sequence with $\sup_r q_r < \infty$, then*

$$x \overset{S_\theta^L}{\sim} y \text{ implies } x \overset{S^L}{\sim} y.$$

Proof. If $\sup_r q_r < \infty$, then there exists $B > 0$ such that $q_r < B$ for all $r \geq 1$. Let $x \overset{S_\theta^L}{\sim} y$, and $\varepsilon > 0$. There exists $R > 0$ such that for every $j \geq R$

$$A_j = \frac{1}{h_j}\left|\left\{k \in I_j : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| < \epsilon.$$

We can also find $K > 0$ such that $A_j < K$ for all $j = 1, 2, \ldots$. Now let n be any integer with $k_{r-1} < n < k_r$, where $r > R$. Then

$$\begin{aligned} \frac{1}{n}\left|\left\{k \leq n : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| &\leq \frac{1}{k_{r-1}}\left|\left\{k \leq k_r : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| \\ &= \frac{1}{k_{r-1}}\left\{\left|\left\{k \in I_1 : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right|\right\} \\ &\quad + \frac{1}{k_{r-1}}\left\{\left|\left\{k \in I_2 : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right|\right\} \\ &\quad + \cdots + \frac{1}{k_{r-1}}\left\{\left|\left\{k \in I_r : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right|\right\} \\ &= \frac{k_1}{k_{r-1}k_1}\left|\left\{k \in I_1 : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| \\ &\quad + \frac{k_2 - k_1}{k_{r-1}(k_2 - k_1)}\left|\left\{k \in I_2 : \left|\frac{x_k}{y_k} - L\right| \geq \epsilon\right\}\right| \end{aligned}$$

$$
\begin{aligned}
&+\cdots+\frac{k_R-k_{R-1}}{k_{r-1}(k_R-k_{R-1})}\left|\left\{k\in I_R:\left|\frac{x_k}{y_k}-L\right|\geq\epsilon\right\}\right| \\
&+\cdots+\frac{k_r-k_{r-1}}{k_{r-1}(k_r-k_{r-1})}\left|\left\{k\in I_r:\left|\frac{x_k}{y_k}-L\right|\geq\epsilon\right\}\right| \\
&=\frac{k_1}{k_{r-1}}A_1+\frac{k_2-k_1}{k_{r-1}}A_2+\cdots+\frac{k_R-k_{R-1}}{k_{r-1}}A_R \\
&+\frac{k_{R+1}-k_R}{k_{r-1}}A_{R+1}+\cdots+\frac{k_r-k_{r-1}}{k_{r-1}}A_r \\
&\leq\left\{\sup_{j\geq 1}A_j\right\}\frac{k_R}{k_{r-1}}+\left\{\sup_{j\geq R}A_j\right\}\frac{k_r-k_R}{k_{r-1}} \\
&\leq K\frac{k_R}{k_{r-1}}+\epsilon B.
\end{aligned}
$$

This completes the proof. □

Theorem 3.4 *Let $\theta=\{k_r\}$ be a lacunary sequence with $1<\inf_r q_r\leq\sup_r q_r<\infty$, then*

$$x\overset{S^L}{\sim}y=x\overset{S_\theta^L}{\sim}y.$$

Proof. The result clearly follows from Theorem 3.2 and 3.3. □

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