On strong double matrix summability via ideals

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Abstract. In this paper, we define some new double sequence spaces by combining the notion of ideal, Orlicz function and nonnegative four dimensional matrix. We make certain investigations on the classes of sequences arising out of this new summability method. In addition, we shall establish inclusion theorems between these spaces and other sequence spaces.

1. Introduction and background

Spaces of strongly summable sequences were studied by Kuttner [10], Maddox [11], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [12] as an extension of the definition of strongly Cesàro summable sequences. Connor [1] further extended this definition to a definition of strong $A$-summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A$-summability, strong $A$-summability with respect to a modulus, and $A$-statistical convergence. Also recently Savas and Patterson [19] extended a few results known in the literature for ordinary (single) sequences to multiply sequences of real and complex numbers. In [14] the notion of convergence for double sequences was presented by A. Pringsheim. Also, in [7] and [16] the four dimensional matrix transformation $(Ax)^{m,n} = \sum_{k,l=1}^{\infty} a_{m,n,k,l} x_{k,l}$ was studied extensively by Hamilton and Robison. In their work and throughout this paper, the four dimensional matrices and double sequences have real-valued entries unless specified otherwise.

On the other hand, ideals were used in [8] to generalize the notion of statistical convergence ([5, 6, 17, 18]). More recent applications of ideals can be seen from ([2, 3, 20]) where more references can be found. In [21], the notion of strong $A^f$–summability with respect to an Orlicz function for single sequence was defined and studied.

Throughout the paper $\mathbb{N}$ will denote the set of all positive integers. A family $I \subset 2^\mathbb{N}$ of subsets of a nonempty set $Y$ is said to be an ideal in $Y$ if (i) $A, B \in I$ implies $A \cup B \in I$; (ii) $A \in I, B \subset A$ implies $B \in I$, while an admissible ideal $I$ of $Y$ further satisfies $\{x\} \in I$ for each $x \in Y$. If $I$ is a proper ideal in $Y$ (i.e., $Y \notin I$, $Y \neq \phi$) then the family of sets $F(I) = \{M \subset Y : \exists A \in I : M = Y \setminus A\}$ is a filter in $Y$. It is called the filter associated with the ideal $I$. Throughout $I$ will stand for a proper non-trivial admissible ideal of $\mathbb{N}$ and $e$ will denote a sequence all of whose elements are 1. Also let $s^\alpha$ denote the set of all double complex or real valued sequences and as usual,

\[ l_\infty^2 = \left\{ x = (x_{k,l}) \in s^\alpha : \|x\| = \sup_{k,l} |x_{k,l}| < \infty \right\}. \]

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In this paper we extend some fundamental theorems of summability theory results from ordinary
(single) sequences spaces to multiply sequence spaces. This will be accomplished by presenting the
following sequence spaces:

\[ x \in s_{\delta}^* : \{ (m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=0}^{\infty} a_{m,n,k,l} F(|x_{k,l}|) \geq \delta \} \in I \text{ for any } \delta > 0 \]

and

\[ x \in s_{\delta'}^* : \{ (m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=0}^{\infty} a_{m,n,k,l} F(|x_{k,l} - L|) \geq \delta \} \in I \text{ for any } \delta > 0 \text{ for some } L \}, \]

where \( F \) is an Orlicz function, and \( A \) is a nonnegative four dimensional matrix. Other implications and
variations will also be presented.

Recall [9] that an Orlicz function is a function \( F : [0, \infty) \to [0, \infty) \) which is continuous, nondecreasing
and convex with \( F(0) = 0, F(x) > 0 \) for \( x > 0 \) and \( F(x) \to \infty \) as \( x \to \infty \). If the convexity of an Orlicz function
\( F \) is replaced by

\[ F(x + y) \leq F(x) + F(y) \]

then it is called a modulus function (see, [12, 15]).

An Orlicz function \( F \) is said to satisfy the \( \Delta_2 \)-condition for all real values of \( u \) if there exists a constant
\( M > 0 \) such that

\[ F(2u) \leq MF(u) \quad (u \geq 0). \]

It can be readily observed that \( F \) satisfies the \( \Delta_2 \)-condition if and only if

\[ F(tu) \leq MtF(u) \]

for all values of \( u \) and \( t > 1 \) [13].

Before continuing with this paper we recall some notations and basic definitions used in this paper. By the convergence in a double sequence we mean the convergence on the Pringsheim sense that is, a
double sequence \( x = (x_{k,l}) \) has Pringsheim limit \( L \) (denoted by \( \text{P-lim} x = L \)) provided that given \( \epsilon > 0 \) there
exists \( N \in \mathbb{N} \) such that \( |x_{k,l} - L| < \epsilon \) whenever \( k, l > N \) [14]. We shall describe such an \( x \) more briefly as \( " \]
P-convergent".

Definition 1.1. Let \( A = (a_{m,n,k,l}) \) denote a four dimensional summability method that maps the complex
double sequences \( x \) into the double sequence \( Ax \) where the \( mn \)-th term to \( Ax \) is as follows:

\[ (Ax)_{m,n} = \sum_{k,l=1}^{\infty} a_{m,n,k,l} x_{k,l}. \]

Such transformation is said to be non-negative if \( a_{m,n,k,l} \) is nonnegative for all \( m, n, k, \) and \( l \).

The notion of regularity for two dimensional matrix transformations was presented by Silverman and
Toeplitz in [22] and [23], respectively. Following Silverman and Toeplitz, Robison and Hamilton presented
the following four dimensional analog of regularity for double sequences in which they both added an
additional assumption of boundedness. This assumption was made because a double sequence which is
P-convergent is not necessarily bounded.

Definition 1.2. The four dimensional matrix \( A \) is said to be \( RH \)-regular if it maps every bounded P-
convergent sequence into a P-convergent sequence with the same P-limit.
In addition to this definition, Robison and Hamilton also presented the following Silverman-Toeplitz type multidimensional characterization of regularity in [7] and [16]:

**Theorem 1.3.** The four dimensional matrix $A$ is RH-regular if and only if

$RH_1$: $P\lim_{m,n}a_{m,n,k,l} = 0$ for each $k$ and $l$;

$RH_2$: $P\lim_{m,n}\sum_{k,l=1}^{\infty}a_{m,n,k,l} = 1$;

$RH_3$: $P\lim_{m,n}\sum_{k=1}^{\infty}|a_{m,n,k,l}| = 0$ for each $l$;

$RH_4$: $P\lim_{m,n}\sum_{l=1}^{\infty}|a_{m,n,k,l}| = 0$ for each $k$;

$RH_5$: $\sum_{k,l=1}^{\infty}|a_{m,n,k,l}|$ is $P$-convergent; and

$RH_6$: there exist positive numbers $A$ and $B$ such that

$\sum_{k,l>0}|a_{m,n,k,l}| < A$.

**Definition 1.4.** Let $A = (a_{m,n,k,l})$ be a non-negative RH-regular four dimensional matrix. A sequence $x = (x_{l,j})$ is said to be $A^l$- double statistically convergent to $L$ if for any $\epsilon > 0$ and $\delta > 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l \in K_2(x-L,\epsilon)} a_{m,n,k,l} \geq \delta \right\} \in I$$

where $K_2(x-L,\epsilon) = \{(k,l) \in \mathbb{N} \times \mathbb{N} : |x_{l,j} - L| \geq \epsilon \}$. In this case we write $x_{l,j} \overset{A^l}{\rightarrow} L$. We denote the class of all $A^l$- double statistically convergent sequences by $S_A^2(I)$.

**2. Main results**

We introduce the following definitions.

**Definition 2.1.** Let $A = (a_{m,n,k,l})$ be a non-negative RH-regular four dimensional matrix and let $I$ be an admissible ideal of $\mathbb{N}$. We define

$$\mathcal{W}^0_0(A)_2 = \left\{ x \in s^2 : \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=0}^{\infty} a_{m,n,k,l} |x_{l,j}| \geq \delta \right\} \in I \text{ for any } \delta > 0 \right\},$$

$$\mathcal{W}^I(A)_2 = \left\{ x \in s^2 : \text{for some } L, x - Le \in \mathcal{W}^0_0(A)_2 \right\}.$$

If $x \in \mathcal{W}^I(A)_2$, we say that $x$ is strongly $A^l$- double summable to $L$. We now introduce the following definitions by using ideals as well as Orlicz function.

Let $A = (a_{m,n,k,l})$ be a non-negative RH-regular four dimensional matrix of real entries, $F$ be an Orlicz function and let $I$ be an admissible ideal of $\mathbb{N}$. We define

$$\mathcal{W}^0_0(A,(F))_2 = \left\{ x \in s^2 : \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=0}^{\infty} a_{m,n,k,l} F(|x_{l,j}|) \geq \delta \right\} \in I \text{ for any } \delta > 0 \right\},$$

$$\mathcal{W}^I(A,(F))_2 = \left\{ x \in s^2 : \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=0}^{\infty} a_{m,n,k,l} F(|x_{l,j} - L|) \geq \delta \right\} \in I, \text{ for some } L \right\}.$$

If $x \in \mathcal{W}^I(A,(F))_2$, we say that $x$ is strongly $A^l$- double summable to $L$ with respect to an Orlicz function $F$.

Let us consider a few special cases of the above sets:

(1) If $F(x) = x$ for all $x \in [0, \infty)$, then we have $\mathcal{W}^I_0(A)_2$ and $\mathcal{W}^I(A)_2$ respectively.
(2) If we take \( A = (C, 1, 1) \), i.e., the double Cesàro matrix, then the above classes of sequences reduce to the following sequence spaces

\[
W^0_0 (F)_2 = \left\{ x \in s' : \left( m, n \in \mathbb{N} \times \mathbb{N} : \frac{1}{n^m} \sum_{k,j=0}^{n,m} a_{m,n,k,j} F(|x_k|) \geq \delta, \text{ for any } \delta > 0 \right) \in I \right\}
\]

and

\[
W^d (F)_2 = \left\{ x \in s' : \left( m, n \in \mathbb{N} \times \mathbb{N} : \frac{1}{n^m} \sum_{k,j=0}^{n,m} a_{m,n,k,j} F(|x_k - L|) \geq \delta \right) \in I, \text{ for some } L \right\}.
\]

(3) Let us consider the following notations and definitions. The double sequence \( \theta_{rs} = (k_r, l_s) \) is called double lacunary if there exist two increasing sequences of integers such that

\[
k_0 = 0, h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty,
\]

\[
l_0 = 0, h_s = l_s - l_{s-1} \to \infty \text{ as } s \to \infty,
\]

and let \( h_{rs} = h_r h_s \), \( \theta_{rs} \) is determine by \( I_{rs} = \{(i, j) : k_{r-1} < i \leq k_r \text{ and } l_{s-1} < j \leq l_s\} \). If we take

\[
a_{r,s,k,j} = \begin{cases} 
\frac{1}{h_r h_s} & \text{if } (k, l) \in I_{rs}; \\
0 & \text{otherwise}. 
\end{cases}
\]

We write

\[
W^0_0 (\theta, F)_2 = \left\{ x \in s' : \left( (r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{k,j \in I_{rs}} F(|x_{k,j}|) \geq \delta \right) \in I, \text{ for any } \delta > 0 \right\}
\]

and

\[
W^d (\theta, F)_2 = \left\{ x \in s' : \left( (r, s) \in \mathbb{N} \times \mathbb{N} : \frac{1}{h_{rs}} \sum_{k,j \in I_{rs}} F(|x_{k,j} - L|) \geq \delta \right) \in I, \text{ for some } L \right\}.
\]

(4) As a final illustration let

\[
a_{i,j,k,l} = \begin{cases} 
\frac{1}{\lambda_i \mu_j} & \text{if } k \in I_i = [i - \lambda_i + 1, i] \text{ and } l \in I_j = [j - \lambda_j + 1, j] \\
0 & \text{otherwise}
\end{cases}
\]

where we shall denote \( \lambda_{i,j} \) by \( \lambda_{i,j} \). Let \( \lambda = (\lambda_i) \) and \( \mu = (\mu_j) \) be two non-decreasing sequences of positive real numbers such that each tending to \( \infty \) and \( \lambda_{i+1} \leq \lambda_i + 1, \lambda_1 = 0 \) and \( \mu_{j+1} \leq \mu_j + 1, \mu_1 = 0 \). Then our definition reduces to the following

\[
W^0_0 (\lambda, F)_2 = \left\{ x \in s' : \left( (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{i,j}} \sum_{k,l \in I_{i,j}} F(|x_{k,l}|) \geq \delta \right) \in I, \text{ for any } \delta > 0 \right\}
\]

\[
W^d (\lambda, F)_2 = \left\{ x \in s' : \left( (i, j) \in \mathbb{N} \times \mathbb{N} : \frac{1}{\lambda_{i,j}} \sum_{k,l \in I_{i,j}} F(|x_{k,l} - L|) \geq \delta \right) \in I, \text{ for some } L \right\}.
\]

It is easy to see that \( W^0_0 (A)_2 \subset W^0_0 (F)_2 \) and \( W^d (A)_2 \subset W^d (F)_2 \) for an Orlicz function \( F \) which satisfies the \( \Delta_2 \)-condition.

We now prove the following result.

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Lemma 2.2. If $A = (a_{m,n,k,l})$ is a non-negative RH-regular four dimensional matrix and $F$ is an Orlicz function which satisfies the $\Delta_2$--condition then

$$W_0^l(A,F)_2 \cap l_{2o}^0$$

is an ideal in $l_{2o}^0$.

Proof. Let $x = (x_{k,l}) \in W_0^l(A,F)_2$ and let $y \in l_{2o}^0$. We show that $xy \in W_0^l(A,F)_2 \cap l_{2o}^0$. Since $y \in l_{2o}^0$, there exists a $M_0 > 1$ such that $\|y\| < M_0$. Then $|x_{k,l}| < M_0 |x_{k,l}|$ for all $(k,l) \in \mathbb{N} \times \mathbb{N}$. Since $F$ is non-decreasing and satisfies the $\Delta_2$--condition, we have

$$F(\epsilon) < F(M_0) = MM_0 \|x_{k,l}\|, \quad (M > 0).$$

Since $x \in W_0^l(A,F)_2$, so

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=0}^{\infty} a_{m,n,k,l} F(|x_{k,l}|) \geq \delta \right\} \in I, \text{ for any } \delta > 0.$$

Hence for $\delta > 0$,

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=0}^{\infty} a_{m,n,k,l} F(|x_{k,l}|) \geq \delta \right\} \subset \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=0}^{\infty} a_{m,n,k,l} MM_0 F(|x_{k,l}|) \geq \delta \right\} = \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=0}^{\infty} a_{m,n,k,l} F(|x_{k,l}|) \geq \frac{\delta}{MM_0} \right\}.$$

Since the set on the right hand side belongs to $I$ so it follows that $xy \in W_0^l(A,F)_2 \cap l_{2o}^0$. \hfill \Box

Lemma 2.3. Let $I$ be an ideal in $l_{2o}^0$ and let $x \in l_{2o}^0$. Then $x$ is in the closure of $I$ in $l_{2o}^0$ if and only if $\chi_{K_2(x,\epsilon)} \in I$ for any $\epsilon > 0$, where $\chi_A$ is the characteristic function of $A$ and $K_2(x,\epsilon) = \{(k,l) \in \mathbb{N} \times \mathbb{N} : |x_{k,l}| \geq \epsilon\}$.

The proof of Lemma 2.2 is straightforward. So we omit it.

Lemma 2.4. If $A = (a_{m,n,k,l})$ is a non-negative RH-regular four dimensional matrix then $W_0^l(A)_2 \cap l_{2o}^0$ is a closed ideal of $l_{2o}^0$.

Proof. Let $x = (x_{k,l})$, $y = (y_{k,l})$ and $x, y \in W_0^l(A)_2 \cap l_{2o}^0$. It is clear that

$$\sum_{k,l=0}^{\infty} a_{m,n,k,l} |x_{k,l} + y_{k,l}| \leq \sum_{k,l=0}^{\infty} a_{m,n,k,l} |x_{k,l}| + \sum_{k,l=0}^{\infty} a_{m,n,k,l} |y_{k,l}|$$

and so for any $\delta > 0$,

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=0}^{\infty} a_{m,n,k,l} |x_{k,l} + y_{k,l}| \geq \delta \right\}$$

$$\subset \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=0}^{\infty} a_{m,n,k,l} |x_{k,l}| \geq \frac{\delta}{2} \right\}$$

$$\cup \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=0}^{\infty} a_{m,n,k,l} |y_{k,l}| \geq \frac{\delta}{2} \right\}.$$
Since \( x, y \in W_0^0(A)_2 \), the sets on the right hand side belong to \( I \) and so is their union. Therefore

\[
\left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,j=0,0}^{\infty} a_{m,n,k,l} |x_{k,j} + y_{k,j}| \geq \delta \right\} \in I
\]

which shows that \( x + y \in W_0^0(A) \cap l_2^\infty \).

Now let \( x \in W_0^0(A) \cap l_2^\infty \) and \( y \in l_2^\infty \). Then there is \( K > 0 \) such that \( |y_{k,j}| \leq K \) for all \((k,l) \in \mathbb{N} \times \mathbb{N}\). Now \( |x_{k,j}| \leq K |x_{k,j}| \) and we have

\[
\left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,j=0,0}^{\infty} a_{m,n,k,l} |x_{k,j}| \geq \delta \right\} \subset \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,j=0,0}^{\infty} a_{m,n,k,l} |x_{k,j}| \geq \frac{\delta}{K} \right\}
\]

for any \( \delta > 0 \). This easily implies that \( xy \in W_0^0(A) \cap l_2^\infty \).

Finally let \((x^{m,n}) \subset W_0^0(A) \cap l_2^\infty \) and let \( x^{m,n} \to x \) in \( l_2^\infty \). We have to show that \( x \in W_0^0(A) \cap l_2^\infty \). Let \( \delta > 0 \) be given. \( (p,q) \in \mathbb{N} \times \mathbb{N} \) such that \( ||x^{p,q} - x||_\infty < \frac{\delta}{2} \).

Now

\[
\sum_{k,j=0,0}^{\infty} a_{m,n,k,l} |x_{k,j}| \leq \sum_{k,j=0,0}^{\infty} a_{m,n,k,l} |x_{k,j}^{p,q} - x_{k,j}| + \sum_{k,j=0,0}^{\infty} a_{m,n,k,l} |x_{k,j}^{p,q}|
\]

\[
\leq \delta \frac{2}{2} + \sum_{k,j=0,0}^{\infty} a_{m,n,k,l} |x_{k,j}^{p,q}|
\]

as \( A = (a_{m,n,k,l}) \) is regular. Evidently then

\[
\left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,j=0,0}^{\infty} a_{m,n,k,l} |x_{k,j}| \geq \delta \right\} \subset \left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,j=0,0}^{\infty} a_{m,n,k,l} |x_{k,j}^{p,q}| \geq \frac{\delta}{2} \right\}
\]

and it follows that \( x \in W_0^0(A) \cap l_2^\infty \). \( \square \)

We now have

**Theorem 2.5.** Let \( x \) be a double bounded sequence, \( F \) be an Orlicz function which satisfies \( \Delta_2 \)-condition and \( A \) be a non-negative regular matrix summability method. Then

\[
W^0(A,F)_2 \cap l_2^\infty = W_0^0(A) \cap l_2^\infty.
\]

**Proof.** We will only show that \( W_0^0(A,F)_2 \cap l_2^\infty = W_0^0(A)_2 \cap l_2^\infty \). Clearly \( W_0^0(A)_2 \cap l_2^\infty \subset W_0^0(A,F)_2 \cap l_2^\infty \) as \( F \) satisfies \( \Delta_2 \)-condition. Write that

\[
\sum_{k,j=0,0}^{\infty} a_{m,n,k,l} F \left( \chi_{K(x-l,e)} (k,l) \right) = \chi_{K(x-e)} (l) \sum_{k,j=0,0}^{\infty} a_{m,n,k,l} \chi_{K(x-l,e)} (k,l)
\]

for all \((m,n) \in \mathbb{N} \times \mathbb{N} \). Let \( x \in W_0^0(A,F)_2 \cap l_2^\infty \) and let \( \epsilon > 0 \) be given. Take the sequence \( y \in l_2^\infty \) by

\[
y_{k,j} = \begin{cases} \frac{1}{\epsilon} \quad & \text{if } |x_{k,j}| \geq \epsilon \\ 0 & \text{otherwise.} \end{cases}
\]

It is easy to see that \( xy = \chi_{K(x,e)} \) which again belongs to \( W_0^0(A,F)_2 \cap l_2^\infty \) (by Lemma 2.1). Then for \( \delta > 0 \),

\[
\left\{ (m, n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,j=0,0}^{\infty} a_{m,n,k,l} F \left( \chi_{K(x,e)} (k,l) \right) \geq \delta \right\} \in I.
\]
But then
\[
\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=0}^{\infty} a_{m,n,k,l} X_K(x_k) (k,l) \geq \delta \right\}
= \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=0}^{\infty} a_{m,n,k,l} F(X_K(x_k) (k,l)) \geq \delta F(1) \right\} \in I.
\]

This shows that \(X_K(x_k) \in W^L_0(A_2) \cap F_\infty\) for any \(\epsilon > 0\) and then it follows from Lemmas 2.2 and 2.3 that

\[x \in W^L_0(A_2) \cap F_\infty, \quad \square\]

**Theorem 2.6.** Let \(A\) be a non-negative RH-regular matrix summability method. Then

(i) \(W^L(A,F)_2 \subset S^2_A(I)\)

(ii) \(S^2_A(I) \cap F_\infty \subset W^L(A,F)_2\) if \(F\) satisfies the \(\Delta_2\)–condition.

**Proof.** (i) Let \(x = (x_{k,l}) \in W^L(A,F)_2\). Then there exists a \(L \in C\) such that for any \(\delta > 0\),

\[\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=0}^{\infty} a_{m,n,k,l} F|x_{k,l} - L| \geq \delta \right\} \subset I.
\]

Now for a fixed \(\epsilon > 0\),
\[
\sum_{k,l=0}^{\infty} a_{m,n,k,l} F|x_{k,l} - L| = \sum_{k,l : |x_{k,l} - L| \geq \epsilon} a_{m,n,k,l} F|x_{k,l} - L| + \sum_{k,l : |x_{k,l} - L| < \epsilon} a_{m,n,k,l} F|x_{k,l} - L| \geq F(\epsilon) \sum_{k,l : |x_{k,l} - L| \geq \epsilon} a_{m,n,k,l}.
\]

Therefore
\[
\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l : |x_{k,l} - L| \geq \epsilon} a_{m,n,k,l} \geq \delta \right\} \subset \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \sum_{k,l=0}^{\infty} a_{m,n,k,l} F|x_{k,l} - L| \geq \delta F(\epsilon) \right\}.
\]

Since the set on the right hand side belongs to \(I\) so it follows that \(x \in S^2_A(I)\).

(ii) If \(x \in S^2_A(I) \cap F_\infty\), then from the definition \(X_K(x_{k,l}, \epsilon) \in W^L_0(A_2) \cap F_\infty\) for every \(\epsilon > 0\) where as usual \(K(x - L, \epsilon) = \{(k,l) \in \mathbb{N} \times \mathbb{N} : |x_{k,l} - L| \geq \epsilon\}\) for some \(L \in C\).

From Lemmas 2.2 and 2.3, \(x \in W^L(A,F)_2 \cap F_\infty\). From Theorem 2.4 it now follows that \(x \in W^L(A,F)_2\), \(\square\)

**Remark 2.7.** Theorem 2.5 presents an improved version of Theorem 3.9 [19] and in a more general form.

**Remark 2.8.** It is easy to see that \(W^L_0(A,F)_2 \cap F_\infty = S^2_A(I) \cap F_\infty\).

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References

[23] O. Toeplitz, Über allgenmeine linear mittelbildungen, Prace Matematyczne Fizyczne (Warsaw) 22 (1911).